

GRAEV METRICS ON FREE PRODUCTS AND HNN EXTENSIONS

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ABSTRACT. We give a construction of two-sided invariant metrics on free products (possibly with amalgamation) of groups with two-sided invariant metrics and, under certain conditions, on HNN extensions of such groups. Our approach is similar to the Graev's construction of metrics on free groups over pointed metric spaces.

1. INTRODUCTION

1.1. History. Back in the 40's in his seminal papers [7, 8] A. Markov came up with a notion of the free topological group over a completely regular (Tychonoff) space. This notion gave birth to a deep and important area in the general theory of topological groups. We highly recommend an excellent overview of free topological groups by O. Sipacheva [9]. Later M. Graev [4] gave another proof of the existence of free topological groups over completely regular spaces. In his approach Graev starts with a pointed metric space (X, x_0, d) and defines in a canonical way a two-sided invariant metric on $F(X \setminus \{x_0\})$ — the free group with bases $X \setminus \{x_0\}$. Moreover, this metric extends the metric d on $X \setminus \{x_0\}$. In modern terms, Graev constructed a functor from the category of pointed metric spaces with Lipschitz maps to the category of groups with two-sided invariant metrics and Lipschitz homomorphisms.

The topology given by the Graev metric on the free group $F(X \setminus \{x_0\})$ is, of course, much weaker than the free topology on $F(X \setminus \{x_0\})$. Since the early 40's a lot of work was done to understand the free topology on free groups, and some of this work shed light onto properties of the Graev metrics.

Graev metrics were used to construct exotic examples of Polish groups (see [1, 6, 11]). For example, the group completion of the free group $F(\mathbb{N}^{\mathbb{N}})$ over the Baire space with the topology given by the Graev metric is an example of a surjectively universal group in the class of Polish groups that admit compatible two-sided invariant metrics (see [6] for the proof).

Once the notion of a free topological group is available, the next step is to construct free products. It was made by Graev himself in [3], where he proves the existence of free products in the category of topological groups. For this he uses, in a clever and unexpected way, Graev metrics on free groups. But this time his approach does not produce a canonical metric on the free product out of metrics on factors.

In this paper we would like to try to push Graev's method from free groups to free products of groups with and without amalgamation. As will be evident from the construction, the natural realm for this approach is the category of groups with two-sided invariant metrics. To be precise, a basic object for us will be an

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abstract group G with a two-sided invariant metric d on it. We recall that G will then automatically be a topological group in the topology given by d . Topological groups that admit a compatible two-sided invariant metric form a very restrictive subclass of the class of all the metrizable topological groups, but it includes compact metrizable and abelian metrizable groups.

1.2. Main results. The paper roughly consists of two parts. In the first part we show the existence of free products of groups with two-sided invariant metrics. Here is a somewhat simplified version of the main theorem.

Theorem (Theorem 5.10). *Let (G_1, d_1) and (G_2, d_2) be groups with two-sided invariant metrics. If $A < G_i$ is a common closed subgroup and $d_1|_A = d_2|_A$, then there is a two-sided invariant metric \underline{d} on the free product with amalgamation $G_1 *_A G_2$ such that $\underline{d}|_{G_i} = d_i$. Moreover, if G_1 and G_2 are separable, then so is $G_1 *_A G_2$.*

Next we address the question of when a two-sided invariant metric can be extended to an HNN extension. We obtain the following results.

Theorem (Theorem 9.1). *Let (G, d) be a tsi group, $\phi : A \rightarrow B$ be a d -isometric isomorphism between the closed subgroups A, B . Let H be the HNN extension of (G, ϕ) in the abstract sense, and let t be the stable letter of the HNN extension. If $\text{diam}(A) \leq K$, then there is a tsi metric \underline{d} on H such that $\underline{d}|_G = d$ and $\underline{d}(t, e) = K$.*

Theorem (Theorem 9.4). *Let G be a SIN metrizable group. Let $\phi : A \rightarrow B$ be a topological isomorphism between two closed subgroups. There exist a SIN metrizable group H and an element $t \in H$ such that $G < H$ is a topological subgroup and $tat^{-1} = \phi(a)$ for all $a \in A$ if and only if there is a compatible tsi metric d on G such that ϕ becomes a d -isometric isomorphism.*

1.3. Notations. We use the following conventions. By an interval we always mean an interval of natural numbers, there will be no intervals of reals in this paper. An interval $\{m, m+1, \dots, n\}$ is denoted by $[m, n]$. For a finite set F of natural numbers $m(F)$ and $M(F)$ denote its minimal and maximal elements respectively. For two sets F_1 and F_2 if $M(F_1) < m(F_2)$, then we say that F_1 is less than F_2 and denote this by $F_1 < F_2$.

A finite set F of natural numbers can be represented uniquely as a union of its maximal sub-intervals, i.e., there are intervals $\{I_k\}_{k=1}^n$ such that

- (i) $F = \bigcup_k I_k$;
- (ii) $M(I_k) + 1 < m(I_{k+1})$ for all $k \in [1, n-1]$.

We refer to such a decomposition of F as to the *family of maximal sub-intervals*.

By a *tree* we mean a connected directed graph without undirected cycles and with a distinguished vertex, which is called the *root* of the tree. For any tree T its root will be denoted by \emptyset . The *height* on a tree T is a function H_T that assigns to a vertex of the tree its graph-theoretic distance to the root. For example $H_T(\emptyset) = 0$ and $H_T(t) = 1$ for all $t \in T \setminus \{\emptyset\}$ such that $(t, \emptyset) \in E(T)$, where $E(T)$ is the set of directed edges of T . We use the word *node* as a synonym for the phrase *vertex of a tree*. We say that a node $s \in T$ is a *predecessor* of $t \in T$, and denote this by $s \prec t$, if there are nodes $s_0, \dots, s_m \in T$ such that $s_0 = t, s_m = s$ and $(s_i, s_{i+1}) \in E(T)$.

For a metric space X its density character, i.e., the smallest cardinality of a dense subset, is denoted by $\chi(X)$.

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2. TRIVIAL WORDS IN AMALGAMS

Let a family $\{G_\lambda\}_{\lambda \in \Lambda}$ of groups be given, where Λ is an index set. Suppose all of the groups contain a subgroup $A \subseteq G_\lambda$, and assume that $G_{\lambda_1} \cap G_{\lambda_2} = A$ for all $\lambda_1 \neq \lambda_2$. Let $G = \bigcup_{\lambda \in \Lambda} G_\lambda$ denote the set-theoretic union of the groups G_λ . The identity element in any group is denoted by e , the ambient group will be evident from the context. Let 0 be a symbol not in Λ . For $g_1, g_2 \in G$ we set $g_1 \cong g_2$ to denote the existence of $\lambda \in \Lambda$ such that $g_1, g_2 \in G_\lambda$. If $g_1 \cong g_2$, we say that g_1 and g_2 are *congruent*. We also define a congruence relation on $\Lambda \cup \{0\}$ by declaring that $x, y \in \Lambda \cup \{0\}$ are congruent if and only if either $x = y$ or at least one of x, y is 0 . This congruence on $\Lambda \cup \{0\}$ is also denoted by \cong .

The free product of the groups G_λ with amalgamation over the subgroup A is denoted by $\coprod_A G_\lambda$. We carefully distinguish words over the alphabet G from elements of the amalgam $\coprod_A G_\lambda$. For that we introduce the following notation. $\text{Words}(G)$ denotes the set of finite nonempty words over the alphabet G . The length of a word $\alpha \in \text{Words}(G)$ is denoted by $|\alpha|$, the concatenation of two words α and β is denoted by $\alpha \frown \beta$, and the i^{th} letter of α is denoted by $\alpha(i)$; in particular, for any $\alpha \in \text{Words}(G)$

$$\alpha = \alpha(1) \frown \alpha(2) \frown \cdots \frown \alpha(|\alpha|).$$

Two words $\alpha, \beta \in \text{Words}(G)$ are said to be *congruent* if $|\alpha| = |\beta|$ and $\alpha(i) \cong \beta(i)$ for all $i \in [1, |\alpha|]$. For technical reasons (to be concrete, for the induction argument in Proposition 2.11) we need the following notion of a labeled word. A *labeled word* is a pair (α, l_α) , where α is a word of length n , and $l_\alpha : [1, n] \rightarrow \Lambda \cup \{0\}$ is a function, called the label of α , such that

$$\alpha(i) \in G_\lambda \setminus A \implies l_\alpha(i) = \lambda$$

for all $i \in [1, n]$.

Example 2.1. Let $\alpha \in \text{Words}(G)$ be any word. There is a canonical label for α given by

$$l_\alpha(i) = \begin{cases} 0 & \text{if } \alpha(i) \in A; \\ \lambda & \text{if } \alpha(i) \in G_\lambda \setminus A. \end{cases}$$

In fact, everywhere, except for the proof of Proposition 2.11, we use this canonical labeling only.

Let α be a word of length n . For a subset $F \subseteq [1, n]$, with $F = \{i_k\}_{k=1}^m$, where $i_1 < i_2 < \cdots < i_m$, set

$$\alpha[F] = \alpha(i_1) \frown \alpha(i_2) \frown \cdots \frown \alpha(i_m).$$

We say that a subset $F \subseteq [1, n]$ is α -congruent if $\alpha(i) \cong \alpha(j)$ for all $i, j \in F$.

There is a natural evaluation map from the set of words $\text{Words}(G)$ over the alphabet G to the amalgam $\coprod_A G_\lambda$ given by the multiplication of letters in the group $\coprod_A G_\lambda$:

$$\alpha \mapsto \alpha(1) \cdot \alpha(2) \cdots \alpha(|\alpha|).$$

This map is denoted by a hat

$$\hat{\cdot} : \text{Words}(G) \rightarrow \coprod_A G_\lambda.$$

Note that this map is obviously surjective. For a word $\alpha \in \text{Words}(G)$ and a subset $F \subseteq [1, |\alpha|]$ we write $\hat{\alpha}[F]$ instead of $\widehat{\alpha[F]}$. We hope this will not confuse the reader too much. A word α is said to be *trivial* if $\hat{\alpha} = e$.

2.1. Structure of trivial words. Elements of the group A will be special for us. Let $\alpha \in \text{Words}(G)$ be a word of length n . We say that its i^{th} letter is *outside of A* if, as the name suggests, $\alpha(i) \notin A$. The *list of external letters* of α is a, possibly empty, sequence $\{i_k\}_{k=1}^m$ such that

- (i) $i_k < i_{k+1}$ for all $k \in [1, m-1]$;
- (ii) $\alpha(i_k) \notin A$ for all $k \in [1, m]$;
- (iii) $\alpha(i) \notin A$ implies $i = i_k$ for some $k \in [1, m]$.

In other words, this is just the increasing list of all the letters in α that are outside of A .

Definition 2.2. Let $\alpha \in \text{Words}(G)$ be a word with the list of external letters $\{i_k\}_{k=1}^m$. The word α is called *alternating* if $\alpha(i_k) \neq \alpha(i_{k+1})$ for all $k \in [1, m-1]$. Note that a word is always alternating if $m \leq 1$. The word α is said to be *reduced* if $\alpha(i) \neq \alpha(i+1)$ for all $i \in [1, |\alpha|-1]$, and it is called a *reduced form* of $f \in \coprod_A G_\lambda$ if additionally $\hat{\alpha} = f$.

The following is a basic fact about free products with amalgamation.

Lemma 2.3. *Let $\alpha \in \text{Words}(G)$ be a reduced word. If $\alpha \neq e$, then $\hat{\alpha} \neq e$.*

It is worth mentioning that if $A \neq \{e\}$, then an element $f \in \coprod_A G_\lambda$ has many different reduced forms (unless $f \in G$, then it has only one). But all these reduced forms have the same length, therefore it is legitimate to talk about the length of an element f itself.

Lemma 2.4. *Any element $f \in \coprod_A G_\lambda$ has a reduced form $\alpha \in \text{Words}(G)$. Moreover, if $\beta \in \text{Words}(G)$ is another reduced form of f , then $|\alpha| = |\beta|$ and $A\alpha(i)A = A\beta(i)A$ for all $i \in [1, |\alpha|]$.*

Proof. The existence of a reduced form of $f \in \coprod_A G_\lambda$ is obvious. Suppose α and β are both reduced forms of f . Set

$$\zeta = \alpha(|\alpha|)^{-1} \cap \cdots \cap \alpha(1)^{-1} \cap \beta(1) \cap \cdots \cap \beta(|\beta|).$$

Since $\hat{\zeta} = e$ and $\zeta \neq e$, by Lemma 2.3 ζ is not reduced. By assumption, α and β were reduced, therefore $\alpha(1) \cong \beta(1)$. We claim that $\alpha(1)^{-1}\beta(1) \in A$. Indeed, if $\alpha(1)^{-1}\beta(1) \notin A$, then the word

$$\xi = \alpha(|\alpha|)^{-1} \cap \cdots \cap \alpha(1)^{-1} \cdot \beta(1) \cap \cdots \cap \beta(|\beta|)$$

is reduced, $\hat{\xi} = e$, and $\xi \neq e$, contradicting Lemma 2.3. So $\alpha(1)^{-1}\beta(1) \in A$, and therefore $\beta(1) = \alpha(1)a_1$ for some $a_1 \in A$ and $A\alpha(1)A = A\beta(1)A$. Now set

$$\alpha_1 = \alpha(2) \cap \cdots \cap \alpha(|\alpha|), \quad \beta_2 = a_1 \cdot \beta(2) \cap \cdots \cap \beta(|\beta|).$$

Since $\hat{\alpha}_1 = \hat{\beta}_1$ and α_1, β_1 are reduced, we can apply the same argument to get $\alpha_1(1) = \beta_1(1)a_2$ for some $a_2 \in A$, whence

$$A\alpha(2)A = A\alpha_1(1)A = A\beta_1(1)A = A\beta(2)A.$$

And we proceed by induction on $|\alpha| + |\beta|$. \square

Lemma 2.5. *Let $f \in \coprod_A G_\lambda$ and $\alpha, \beta \in \text{Words}(G)$ be given. If α is a reduced form of f , $|\alpha| = |\beta|$ and $\hat{\alpha} = \hat{\beta}$, then β is a reduced form of f .*

Proof. If β is not a reduced form of f , we perform cancellations in β and get a reduced word β_1 such that $\hat{\beta}_1 = \hat{\beta}$ and $|\beta_1| < |\beta|$. By Lemma 2.4 we have $|\beta_1| = |\alpha|$, contradicting $|\beta| = |\alpha|$. Hence β is reduced. \square

Lemma 2.6. *If α is an alternating word with a nonempty list of external letters, then $\hat{\alpha} \neq e$.*

Proof. Let $\{i_k\}_{k=1}^m$ be the list of external letters of α . For $k \in [2, m-1]$ set

$$\begin{aligned}\xi_1 &= \alpha(1) \cdots \alpha(i_2 - 1), \\ \xi_k &= \alpha(i_k) \cdot \alpha(i_k + 1) \cdots \alpha(i_{k+1} - 1), \\ \xi_m &= \alpha(i_m) \cdot \alpha(i_m + 1) \cdots \alpha(n),\end{aligned}$$

and put

$$\xi = \xi_1 \frown \cdots \frown \xi_m.$$

Then $\hat{\xi} = \hat{\alpha}$, $\xi \neq e$ (since $\xi_i \neq e$ for all $i \in [1, m]$), and, as one easily checks, ξ is reduced. An application of Lemma 2.3 finishes the proof. \square

Lemma 2.7. *If ζ is a trivial word of length n with a nonempty list of external letters, then there is an interval $I \subseteq [1, n]$ such that*

- (i) $\hat{\zeta}[I] \in A$;
- (ii) I is ζ -congruent;
- (iii) $\zeta(m(I)), \zeta(M(I)) \notin A$.

Proof. Let $\{i_k\}_{k=1}^m$ be the list of external letters. For all $k \in [1, m]$ define m_k and M_k by

$$\begin{aligned}m_k &= \min\{j \in [1, k] : [i_j, i_k] \text{ is } \zeta\text{-congruent}\}, \\ M_k &= \max\{j \in [k, m] : [i_k, i_j] \text{ is } \zeta\text{-congruent}\}.\end{aligned}$$

Set $I_k = [m_k, M_k]$, and note that for $k, l \in [1, m]$

$$I_k \cap I_l \neq \emptyset \implies I_l = I_k.$$

Let I_{k_1}, \dots, I_{k_p} be a list of all the distinct intervals I_{k_i} . Then $\{I_{k_i}\}_{i=1}^p$ are pairwise disjoint. Note that each of I_{k_i} satisfies items (ii) and (iii). To prove the lemma it is enough to show that for some $i \in [1, p]$ the corresponding I_{k_i} satisfies also item (i). Suppose this is false and $\hat{\zeta}[I_{k_i}] \notin A$ for all $i \in [1, p]$. Set $\xi_i = \hat{\zeta}[I_{k_i}]$ and

$$\begin{aligned}\xi &= \zeta(1) \frown \cdots \frown \zeta(m(I_{k_1}) - 1) \frown \xi_1 \frown \zeta(M(I_{k_1}) + 1) \frown \cdots \\ &\quad \cdots \frown \zeta(m(I_{k_2}) - 1) \frown \xi_2 \frown \zeta(M(I_{k_2}) + 1) \frown \cdots \\ &\quad \cdots \frown \zeta(m(I_{k_p}) - 1) \frown \xi_p \frown \zeta(M(I_{k_p}) + 1) \frown \cdots \frown \zeta(n).\end{aligned}$$

Then, of course, $\hat{\xi} = \hat{\zeta} = e$ and ξ is alternating by the choice of $\{I_{k_i}\}$. By Lemma 2.6 the word ξ is non-trivial, which is a contradiction. \square

Lemma 2.8. *If (ζ, l_ζ) is a trivial labeled word of length n with a nonempty list of external letters, then there is an interval $I \subseteq [1, n]$ such that*

- (i) $\hat{\zeta}[I] \in A$;
- (ii) I is ζ -congruent;
- (iii) $\zeta(i) \notin A$ for some $i \in I$;
- (iv) if $m(I) > 1$, then $l_\zeta(m(I) - 1) \neq 0$; if $M(I) < n$, then $l_\zeta(M(I) + 1) \neq 0$;
- (v) if $\zeta(m(I)) \in A$, then $l_\zeta(m(I)) = 0$; if $\zeta(M(I)) \in A$, then $l_\zeta(M(I)) = 0$.

Proof. We start by applying Lemma 2.7 to the word ζ . This Lemma gives as an output an interval $J \subseteq [1, n]$. We will now enlarge this interval as follows. If $l_\zeta(i) = 0$ for all $i \in [1, m(J) - 1]$, then set $j_l = 1$. If there is some $i < m(J)$ such that $l_\zeta(i) \neq 0$, then let $j \in [1, m(J) - 1]$ be maximal such that $l_\zeta(j) \neq 0$ and set $j_l = j + 1$. Similarly, if $l_\zeta(i) = 0$ for all $i \in [M(J) + 1, n]$, then set $j_r = n$. If there is some $i > M(J)$ such that $l_\zeta(i) \neq 0$, then let $j \in [M(J) + 1, n]$ be minimal such that $l_\zeta(j) \neq 0$ and set $j_r = j - 1$. Define

$$I = J \cup [j_l, m(J)] \cup [M(J), j_r] = [j_l, j_r].$$

We claim that I satisfies the assumptions. Note that $J \subseteq I$ and $I \setminus J \subseteq A$, so (i), (ii) and (iii) follow from items (i), (ii) and (iii) of Lemma 2.7. Items (iv) and (v) follow from the choice of j_l and j_r and from item (iii) of Lemma 2.7. \square

Definition 2.9. Let (ζ, l_ζ) be a trivial labeled word of length n , and let T be a tree. Suppose that to each node $t \in T$ an interval $I_t \subseteq [1, n]$ is assigned. Set $R_t = I_t \setminus \bigcup_{t' \prec t} I_{t'}$. The tree T together with the assignment $t \mapsto I_t$ is called an *evaluation tree* for (ζ, l_ζ) if for all $s, t \in T$ the following holds:

- (i) $I_\emptyset = [1, n]$;
- (ii) $\hat{\zeta}[I_t] \in A$;
- (iii) if $t \neq \emptyset$ and $m(I_t) \in A$, then $l_\zeta(m(I_t)) = 0$; if $t \neq \emptyset$ and $M(I_t) \in A$, then $l_\zeta(M(I_t)) = 0$;
- (iv) if $H(t) \leq H(s)$ and $I_s \cap I_t \neq \emptyset$, then $s \prec t$ or $s = t$;
- (v) if $s \prec t$ and $t \neq \emptyset$, then

$$m(I_t) < m(I_s) \leq M(I_s) < M(I_t);$$

- (vi) $\zeta(i) \cong \zeta(j)$ for all $i, j \in R_t$;

An evaluation tree T is called *balanced* if additionally the following two conditions hold:

- (vii) if $T_\zeta \neq \{\emptyset\}$, then for any $t \in T_\zeta$ if R_t is written as a disjoint union of maximal sub-intervals $\{I_j\}_{j=1}^k$, then for any j there is $i \in I_j$ such that $l_\zeta(i) \neq 0$;
- (viii) if $s \prec t$, then

$$m(I_s) - 1 \in R_t \implies l_\zeta(m(I_s) - 1) \neq 0;$$

$$M(I_s) + 1 \in R_t \implies l_\zeta(M(I_s) + 1) \neq 0.$$

Remark 2.10. Note that if $\zeta \in \text{Words}(G)$ is a trivial word with the canonical label as in Example 2.1, then item (iii) in the definition of an evaluation tree is vacuous.

Proposition 2.11. *Any trivial labeled word (ζ, l_ζ) has a balanced evaluation tree.*

Proof. We prove the proposition by induction on the cardinality of the list of external letters of ζ . Suppose first that the list is empty, and $\zeta(i) \in A$ for all $i \in [1, n]$.

Set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. It is easy to check that all the conditions are satisfied, and T_ζ is a balanced evaluation tree for (ζ, l_ζ) .

From now on we assume there is $i \in [1, n]$ such that $\zeta(i) \notin A$. Apply Lemma 2.8 to (ζ, l_ζ) and let I be the interval granted by this lemma. Set $\lambda_0 = l_\zeta(i)$ for some (equivalently, any) $i \in I$ such that $\zeta(i) \notin A$. Note that $\lambda_0 \neq 0$. Let $m = |I|$ be the length of I . If $m = n$, then we set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. Similarly to the base of induction this tree is a balanced evaluation tree for (ζ, l_ζ) . From now on we assume that $m < n$. We define the word ξ of length $n - m + 1$ as follows. Set

$$\xi(i) = \begin{cases} \zeta(i) & \text{if } i < m(I) \\ \hat{\zeta}[I] & \text{if } i = m(I) \\ \zeta(i + m - 1) & \text{if } i > m(I). \end{cases}$$

Define the label for ξ to be

$$l_\xi(i) = \begin{cases} l_\zeta(i) & \text{if } i < m(I) \\ \lambda_0 & \text{if } i = m(I) \\ l_\zeta(i + m - 1) & \text{if } i > m(I). \end{cases}$$

We claim that

$$|\{i \in [1, |\xi|] : \xi(i) \notin A\}| < |\{i \in [1, n] : \zeta(i) \notin A\}|.$$

Indeed, by the construction $\zeta[I]$ has at least one letter (in fact, at least two letters) not from A .

By inductive assumption applied to the labeled word (ξ, l_ξ) , there is a balanced evaluation tree T_ξ with intervals $J_t \subseteq [1, |\xi|]$ for $t \in T_\xi$. Since $J_\emptyset = [1, |\xi|]$, there is at least one $t \in T_\xi$ (namely $t = \emptyset$) such that the interval J_t contains $m(I)$. By item (iv) there is the smallest node $t_0 \in T_\xi$ such that $m(I) \in J_{t_0}$.

We define T_ζ to be $T_\xi \cup \{s_0\}$, where s_0 is a new predecessor of t_0 , i. e. , $s_0 \prec t_0$. For $t \in T_\xi$ set

$$I_t = \begin{cases} [m(J_t), M(J_t)] & \text{if } M(J_t) < m(I); \\ [m(J_t), M(J_t) + m - 1] & \text{if } m(J_t) \leq m(I) \leq M(J_t); \\ [m(J_t) + m - 1, M(J_t) + m - 1] & \text{if } m(I) < m(J_t); \end{cases}$$

and

$$I_{s_0} = [m(I), M(I)].$$

We claim that such a tree T_ζ with such an assignment of intervals I_t is a balanced evaluation tree for (ζ, l_ζ) .

- (i) Since $J_\emptyset = [1, |\xi|]$, it follows that $I_\emptyset = [1, n]$.
- (ii) For any $t \in T_\xi$ one has $\hat{\xi}[J_t] = \hat{\zeta}[I_t]$. Also, $\hat{\zeta}[I_{s_0}] \in A$ by item (i) of Lemma 2.8.
- (iii) Since $\xi(m(I)) \in A$ and $l_\xi(m(I)) = \lambda_0 \neq 0$, by inductive hypothesis $m(I_t) \neq m(I)$ and $M(I_t) \neq m(I)$ for all $t \in T_\xi \setminus \{\emptyset\}$. Therefore $l_\xi(m(J_t)) = l_\zeta(m(I_t))$, $l_\xi(M(J_t)) = l_\zeta(M(I_t))$ for all $t \in T_\xi \setminus \{\emptyset\}$. Thus for $t \neq s_0$ the item follows from the inductive hypothesis, and for $t = s_0$ it follows from item (v) of Lemma 2.8.
- (iv) Follows from the inductive hypothesis and the definition of s_0 .
- (v) It follows from the inductive hypothesis that this item is satisfied for all $s, t \in T_\xi$. We need to consider the case $s = s_0, t = t_0$ only. By item (iii) of the definition of an evaluation tree, and since $l_\xi(m(I)) = \lambda_0 \neq 0$, it follows that if $t_0 \neq \emptyset$, then $m(I_{t_0}) < m(I_{s_0})$ and $M(I_{s_0}) < M(I_{t_0})$.

(vi) Follows easily from the inductive hypothesis and item (ii) of Lemma 2.8.

Thus T_ζ is an evaluation tree for (ζ, l_ζ) . It remains to check that it is balanced.

(vii) For $t \in T_\xi \setminus \{t_0\}$ the maximal sub-intervals of $J_t \setminus \cup_{s \prec t} J_s$ naturally correspond to the maximal sub-intervals of $I_t \setminus \cup_{s \prec t} I_s$, and hence for such a t the item follows from the inductive hypothesis. For $t = s_0$ the item follows from item (iii) of Lemma 2.8. The remaining case $t = t_0$ follows from item (iv) of Lemma 2.8.

(viii) Again, for $s \neq s_0$ this item follows from the inductive hypothesis and for $s = s_0, t = t_0$ follows from item (iv) of Lemma 2.8. \square

If ζ is just a word with no labeling, then we canonically associate a label to it by declaring $l_\zeta(i) = 0$ if and only if $\zeta(i) \in A$ (as in Example 2.1).

From now on we view all trivial words as labeled words with the canonical labeling.

Definition 2.12. A trivial word $\zeta \in \text{Words}(G)$ of length n is called *slim* if there exists an evaluation tree T_ζ such that $\hat{\zeta}[I_t] = e$ for all $t \in T_\zeta$; such a tree is then called a *slim* evaluation tree. We say that ζ is *simple* if it is slim and $\zeta(i) \in A$ implies $\zeta(i) = e$ for all $i \in [1, n]$.

Definition 2.13. Let $f \in \coprod_A G_\lambda$. A pair of words (α, ζ) is called an *f-pair* if $|\alpha| = |\zeta|$ and $\hat{\alpha} = f, \hat{\zeta} = e$. An *f-pair* (α, ζ) is said to be a *congruent f-pair* if α is congruent to ζ . An *f-pair* (α, ζ) is called *slim* if it is congruent and ζ is slim. It is called *simple* if it is congruent and ζ is simple.

For a congruent pair (α, β) of length n we define the notions of right and left transfers. Let $a \in A$ and $i \in [1, n-1]$ be given. The *right (a, i)-transfer* of (α, β) is the pair $\text{RTran}(\alpha, \beta; a, i) = (\gamma, \delta)$ defined as follows:

$$(\gamma(j), \delta(j)) = \begin{cases} (\alpha(j), \beta(j)) & \text{if } j \notin \{i, i+1\}; \\ (\alpha(i)a^{-1}, \beta(i)a^{-1}) & \text{if } j = i; \\ (a\alpha(i+1), a\beta(i+1)) & \text{if } j = i+1. \end{cases}$$

For $a \in A$ and $i \in [2, n]$ the *left (a, i)-transfer* of (α, β) is denoted by $\text{LTran}(\alpha, \beta; a, i) = (\gamma, \delta)$ and is defined as

$$(\gamma(j), \delta(j)) = \begin{cases} (\alpha(j), \beta(j)) & \text{if } j \notin \{i-1, i\}; \\ (a^{-1}\alpha(i), a^{-1}\beta(i)) & \text{if } j = i; \\ (\alpha(i-1)a, \beta(i-1)a) & \text{if } j = i-1. \end{cases}$$

We will typically have specific sequences of transfers, so it is convenient to make the following definition. Let (α, ζ) be a congruent pair of words of length n . In all the applications ζ will be a trivial word. Let $\{I_k\}_{k=1}^m$ be a sequence of intervals such that:

- (1) $I_k \subseteq [1, n]$;
- (2) $I_k < I_{k+1}$ for all $k \in [1, m-1]$;
- (3) $\hat{\zeta}[I_k] \in A$ for all $k \in [1, m]$;
- (4) $M(I_m) < n$.

Such a sequence is called *right transfer admissible*. If together with items (1) – (3) the following is satisfied

- (4') $m(I_1) > 1$,

then the sequence $\{I_k\}_{k=1}^m$ is called *left transfer admissible*.

Let $\{I_k\}_{k=1}^m$ be a right transfer admissible sequence of intervals. Define inductively words (β_k, ξ_k) by setting $(\beta_0, \xi_0) = (\alpha, \zeta)$ and

$$(\beta_{k+1}, \xi_{k+1}) = \text{RTran}(\beta_k, \xi_k; \hat{\xi}_k[I_{k+1}], M(I_{k+1})).$$

We have to show that the right-hand side is well-defined, i.e., that $\hat{\xi}_k[I_{k+1}] \in A$. For the first step of the construction we have $\hat{\xi}_0[I_1] = \hat{\zeta}[I_1] \in A$, because the sequence is right transfer admissible. Suppose we have proved that $\hat{\xi}_{k-1}[I_k] \in A$. There are two cases: either $M(I_k) + 1 = m(I_{k+1})$, and then

$$\hat{\xi}_k[I_{k+1}] = (\hat{\xi}_{k-1}[I_k]) \cdot \hat{\zeta}[I_{k+1}],$$

or $M(I_k) + 1 < m(I_{k+1})$, and then $\hat{\xi}_k[I_{k+1}] = \hat{\zeta}[I_{k+1}]$. In both cases we get $\hat{\xi}_k[I_{k+1}] \in A$.

By definition, the right $\{I_k\}$ -transfer of (α, ζ) is the pair (β_m, ξ_m) .

The left transfer is defined similarly, but with one extra change: we apply left transfers in the decreasing order from I_m to I_1 . Here is a formal definition. For a left admissible sequence of intervals $\{I_k\}_{k=1}^m$ set inductively $(\beta_0, \xi_0) = (\alpha, \zeta)$ and

$$(\beta_{k+1}, \xi_{k+1}) = \text{LTran}(\beta_k, \xi_k; \hat{\xi}_k[I_{m-k}], m(I_{m-k})).$$

Similarly to the case of the right transfer one shows that the right-hand side in the above construction is well-defined. By definition, the left $\{I_k\}$ -transfer of (α, ζ) is the pair (β_m, ξ_m) .

This notion of transfer, though a bit technical, will be crucial in some reductions in the next section. The following lemma establishes basic properties of the transfer operation with respect to the earlier notion of the evaluation tree.

Lemma 2.14. *Let (α, ζ) be a congruent f -pair of length n and let T_ζ be a [balanced] evaluation tree for ζ . Let $\{I_k\}_{k=1}^m$ be a right [left] transfer admissible sequence of intervals. Let (β, ξ) be the right [left] $\{I_k\}$ -transfer of (α, ζ) . Then*

- (i) $|\beta| = n = |\xi|$;
- (ii) (β, ξ) is a congruent f -pair;
- (iii) T_ξ is a [balanced] evaluation tree for ξ .
- (iv) $\xi(i) = \zeta(i)$ for all $i \notin \{M(I_k), M(I_{k+1}) : k \in [1, m]\}$ for the right transfer and for all $i \notin \{m(I_k), m(I_{k-1}) - 1 : k \in [1, m]\}$ in the case of the left transfer;
- (v) $\hat{\xi}[I_k] = e$ for all $k \in [1, m]$.

Proof. Items (i), (ii), and (iv) are trivial; item (iii) follows easily from the observation that $\xi(i) \in A$ if and only if $\zeta(i) \in A$. For item (v) let ξ_k be as in the definition of the $\{I_k\}$ -transfer. Suppose for definiteness that we are in the case of the right transfer. Then $\hat{\xi}_k[I_k] = e$ by construction and also $\xi_{k+1}[I_j] = \xi_k[I_j]$ for all $j \in [1, k]$. The lemma follows. \square

We will later need another operation on words, we call it symmetrization. Here is the definition.

Definition 2.15. Let (α, ζ) be a slim f -pair with a slim evaluation tree T_ζ . Let $t \in T_\zeta$ and $\{i_k\}_{k=1}^m \subseteq R_t$ be a list such that

- (i) $i_k < i_{k+1}$ for $k \in [1, m-1]$;
- (ii) if $\zeta(i) \neq e$ for some $i \in R_t$, then $i = i_k$ for some $k \in [1, m]$;
- (iii) $\alpha(i_k) \cong \alpha(i_l)$ for all $k, l \in [1, m]$.

Such a list is called *symmetrization admissible*. For $j_0 \in \{i_k\}_{k=1}^m$ let k_0 be such that $j_0 = i_{k_0}$ and define a *symmetrization* $\text{Sym}(\alpha, \zeta; j_0, \{i_k\}_{k=1}^m)$ of ζ to be the word ξ such that

$$\xi(i) = \begin{cases} \zeta(i) & \text{if } i \neq i_p \text{ for all } p \in [1, m]; \\ \alpha(i) & \text{if } i \in \{i_k\}_{k=1}^m \setminus \{j_0\}; \\ \alpha(i_{k_0-1})^{-1} \dots \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \dots \alpha(i_{k_0+1})^{-1} & \text{if } i = j_0. \end{cases}$$

If $m = 1$, the above definition does not make sense, so we set that in this case $\text{Sym}(\alpha, \zeta; i_1, i_1) = \zeta$.

Lemma 2.16. *Let (α, ζ) be a slim f -pair with a slim evaluation tree T_ζ . Let $t \in T_\zeta$, and let $\{i_k\}_{k=1}^m \subseteq R_t$ be a symmetrization admissible list. Fix some $j_0 \in \{i_k\}_{k=1}^m$. If ξ is the symmetrization $\text{Sym}(\alpha, \zeta; j_0, \{i_k\}_{k=1}^m)$ of ζ , then (α, ξ) is a slim f -pair and T_ξ is a slim evaluation tree for ξ with the same assignment of intervals $t \mapsto I_t$.*

Proof. The only non-trivial part in the lemma is to show that $\hat{\xi}[I_t] = e$. This follows from the facts that $\hat{\zeta}[I_s] = e$ for all $s \prec t$ (because T_ζ is slim) and that $\zeta(i) = e$ for all $i \in R_t \setminus \{i_1, \dots, i_m\}$ (by the definition of the symmetrization admissible list). \square

3. GROUPS WITH TWO-SIDED INVARIANT METRICS

In this section we would like to recall some facts from the theory of groups with two-sided invariant metrics. The reader can consult [2] for the details.

Definition 3.1. A metric d on a group G is called two-sided invariant if

$$d(gf_1, gf_2) = d(f_1, f_2) = d(f_1g, f_2g)$$

for all $g, f_1, f_2 \in G$. A tsi group is a pair (G, d) , where G is a group and d is a two-sided invariant metric on G ; tsi stands for two-sided invariant.

Proposition 3.2. *If (G, d) be a tsi group, then G is a topological group in the topology of the metric d .*

Proposition 3.3. *Let d be a left invariant metric on the group G .*

(i) *If for all $g_1, g_2, f_1, f_2 \in G$*

$$d(g_1g_2, f_1f_2) \leq d(g_1, f_1) + d(g_2, f_2),$$

then d is two-sided invariant;

(ii) *If d is two-sided invariant, then for all $g_1, \dots, g_k, f_1, \dots, f_k \in G$*

$$d(g_1 \cdots g_k, f_1 \cdots f_k) \leq \sum_{i=1}^k d(g_i, f_i).$$

Because of Proposition 3.2 we choose to speak not about topological groups that admit a compatible two-sided invariant metric, but rather about abstract groups with a two-sided invariant metric. Note that the class of metrizable groups that admit a compatible two-sided invariant metric is very small, but it includes two important subclasses: abelian and compact metrizable groups.

The class of tsi groups is closed under taking factors by closed normal subgroups, and, moreover, there is a canonical metric on the factor.

Proposition 3.4. *If (G, d) is a tsi group and $N < G$ is a closed normal subgroup, then the function*

$$d_0(g_1N, g_2N) = \inf\{d(g_1h_1, g_2h_2) : h_1, h_2 \in N\}$$

is a two-sided invariant metric on the factor group G/N and the factor map $\pi : G \rightarrow G/N$ is a 1-Lipschitz surjection from (G, d) onto $(G/N, d_0)$.

The metric d_0 is called the *factor metric*.

Proposition 3.5. *Let (G, d) be a tsi group. Let (\overline{G}, d) be the completion of G as a metric space; the extension of the metric d on G to the completion \overline{G} is again denoted by d . There is a unique extension of group operation from G to \overline{G} . This extension turns (\overline{G}, d) into a tsi group.*

This proposition states that for tsi groups metric and group completions are the same.

4. GRAEV METRIC GROUPS

Before going into the details of the construction of Graev metrics on free products we would like to recall the definition of the Graev metrics on free groups. The reader may consult [4], [1], [2] or [6] for the details and proofs.

Classically one starts with a pointed metric space (X, e, d) , where d is a metric and $e \in X$ is a distinguished point. Take another copy of this space, denote it by (X^{-1}, d) , and its elements are the formal inverses of the elements in X with the agreement $e^{-1} = e$ and $X \cap X^{-1} = \{e\}$. Then X^{-1} is also a metric space and we can amalgamate (X, d) and (X^{-1}, d) over the point e . Denote the resulting space by (\overline{X}, e, d) . Equivalently, $\overline{X} = X \cup X^{-1}$, and for all $x, y \in X$

$$d(x^{-1}, y^{-1}) = d(x, y), \quad d(x, y^{-1}) = d(x, e) + d(e, y).$$

With the set \overline{X} we associate two objects: the set of *nonempty* words $\text{Words}(\overline{X})$ over the alphabet \overline{X} and the free group $F(X)$ over the basis X . There is a small issue with the second object. We want e to be the identity element of this group rather than an element of the basis. In other words, we formally have to write $F(X \setminus \{e\})$, but we adopt the convention that given a pointed metric space (X, e, d) , in $F(X)$ the letter $e \in X$ is interpreted as the identity element. The inverse operation in $F(X)$ naturally extends the inverse operation on \overline{X} . We have a natural map

$$\hat{} : \text{Words}(\overline{X}) \rightarrow F(X),$$

for $u \in \text{Words}(\overline{X})$ its image \hat{u} is just the reduced form of u . For a word $u \in \text{Words}(\overline{X})$ its length is denoted by $|u|$ and its i^{th} letter is denoted by $u(i)$. For two words $u, v \in \text{Words}(\overline{X})$ of the same length n we define a function

$$\rho(u, v) = \sum_{i=1}^n d(u(i), v(i)).$$

And finally, we define a metric \underline{d} by

$$\underline{d}(f, g) = \inf\{\rho(u, v) : |u| = |v| \text{ and } \hat{u} = f, \hat{v} = g\}.$$

A theorem of Graev [4] states that \underline{d} is indeed a two-sided invariant metric on $F(X)$, and moreover, it extends the metric d on the amalgam \overline{X} . It is straightforward to see that \underline{d} is a two-sided invariant *pseudo*-metric and the hard part of the Graev's

theorem is to show that it assigns a non-zero distance to distinct elements. Graev showed this by proving some restrictions on u and v in the infimum in the definition of d . Later there were some further results in this direction, which greatly improved our understanding of the Graev metrics, most notably [10] and [1]. In our presentation we follow L. Ding and S. Gao [1].

Definition 4.1. Let I be an interval of natural numbers. A bijection $\theta : I \rightarrow I$ is called a *match* if

- (i) $\theta \circ \theta = \text{id}$;
- (ii) there are no $i, j \in I$ such that $i < j < \theta(i) < \theta(j)$.

Definition 4.2. Let $w \in \text{Words}(\overline{X})$ be a word of length n , let θ be a match on $[1, n]$. A word w^θ has length n and is defined as

$$w^\theta(i) = \begin{cases} e & \text{if } \theta(i) = i; \\ w(i) & \text{if } \theta(i) > i; \\ w(\theta(i))^{-1} & \text{if } \theta(i) < i. \end{cases}$$

It is not hard to check that for any word w and any match θ on $[1, |w|]$ the word w^θ is trivial, i.e. $\widehat{w^\theta} = e$.

Theorem 4.3 (Ding–Gao). *If $f \in F(X)$ and $w \in \text{Words}(\overline{X})$ is the reduced form of f , then*

$$\underline{d}(f, e) = \min \{ \rho(w, w^\theta) : \theta \text{ is a match on } [1, |w|] \}.$$

Here are some of the properties of the Graev metrics. They are easy consequences of the definition of the Graev metric and Theorem 4.3.

Proposition 4.4. *Let (X, e, d) be a pointed metric space, and let \underline{d} be the Graev metric on $F(X)$.*

- (i) *If (T, d_T) is a tsi group and $\phi : X \rightarrow T$ is a K -Lipschitz map such that $\phi(e) = e$, then this map extends uniquely to a K -Lipschitz homomorphism $\phi : F(X) \rightarrow T$.*
- (ii) *If $Y \subseteq X$, $e \in Y$ is a pointed subspace of X with the induced metric, then the natural embedding $i : Y \rightarrow X$ extends uniquely to an isometric embedding*

$$i : F(Y) \rightarrow F(X).$$

Moreover, if Y is closed in X , then $F(Y)$ is closed in $F(X)$.

- (iii) *If δ is any tsi metric $F(X)$ that extends d , i.e., if $d(x_1, x_2) = \delta(x_1, x_2)$ for all $x_1, x_2 \in X$, then $\delta(u_1, u_2) \leq \underline{d}(u_1, u_2)$ for all $u_1, u_2 \in F(X)$. In other words, \underline{d} is maximal among all the tsi metrics that extend d .*
- (iv) *If $X \neq \{e\}$, then*

$$\chi(F(X)) = \max\{\aleph_0, \chi(X)\}.$$

In particular, if X is separable, then so is $F(X)$.

4.1. Free groups over metric groups. In this subsection we prove a technical result that will be used later in Section 6.

Suppose X is itself a group and $e \in X$ is the identity element of that group. Let \circ denote the multiplication operation on X , and let x^\dagger denote the group inverse of

an element $x \in X$. Suppose also that d is a two sided invariant metric on X . For $u \in \text{Words}(\overline{X})$ define a word u^\sharp by

$$u^\sharp(i) = \begin{cases} u(i) & \text{if } u(i) \in X; \\ (u(i)^{-1})^\dagger & \text{if } u(i) \in X^{-1}. \end{cases}$$

For $h \in F(X)$ let $h^\sharp = \widehat{w^\sharp}$, where w is the reduced form of h .

Proposition 4.5. *Let $f \in F(X)$, and let w be the reduced form of f . If $w \in \text{Words}(X)$, then for any $h \in F(X)$*

$$\underline{d}(fh, e) \geq \underline{d}(fh^\sharp, e).$$

Proof. Suppose $w \in \text{Words}(X)$ and fix an $h \in F(X)$. Let $u \in \text{Words}(\overline{X})$ be the reduced form of h . It is enough to show that

$$\rho(w \frown u, (w \frown u)^\theta) \geq \rho(w \frown u^\sharp, (w \frown u^\sharp)^\theta)$$

for any match θ on $[1, |w| + |u|]$. This follows from the following inequalities:

- if $x, y \in X^{-1}$, then by the two-sided invariance of the metric d

$$d(x, y) = d(x^{-1}, y^{-1}) = d((x^{-1})^\dagger, (y^{-1})^\dagger);$$

- if $x \in X^{-1}$ and $y \in X$, then by the two-sided invariance of the metric d

$$\begin{aligned} d(x, y) &= d(x, e) + d(e, y) = d(x^{-1}, e) + d(e, y) = \\ &= d((x^{-1})^\dagger, e) + d(e, y) \geq d((x^{-1})^\dagger, y). \end{aligned}$$

Thus $\underline{d}(fh, e) \geq \underline{d}(fh^\sharp, e)$. □

5. METRICS ON AMALGAMS

5.1. Basic set up. Let (G_λ, d_λ) be a family of tsi groups, $A < G_\lambda$ be a common closed subgroup, $G_{\lambda_1} \cap G_{\lambda_2} = A$, and assume additionally that the metrics $\{d_\lambda\}$ agree on A :

$$d_{\lambda_1}(a_1, a_2) = d_{\lambda_2}(a_1, a_2) \quad \text{for all } a_1, a_2 \in A \text{ and all } \lambda_1, \lambda_2 \in \Lambda.$$

Our main goal is to define a metric on the free product of G_λ with amalgamation over A that extends all the metrics d_λ . It will be an analog of the Graev metrics on free groups.

First of all, let d denote the amalgam metric on $G = \bigcup_\lambda G_\lambda$ given by

$$d(f_1, f_2) = \begin{cases} d_\lambda(f_1, f_2) & \text{if } f_1, f_2 \in G_\lambda \text{ for some } \lambda \in \Lambda; \\ \inf_{a \in A} \{d_{\lambda_1}(f_1, a) + d_{\lambda_2}(a, f_2)\} & \text{if } f_1 \in G_{\lambda_1}, f_2 \in G_{\lambda_2} \text{ for } \lambda_1 \neq \lambda_2. \end{cases}$$

If α_1 and α_2 are two words in $\text{Words}(G)$ of the same length n , then the value $\rho(\alpha_1, \alpha_2)$ is defined by

$$\rho(\alpha_1, \alpha_2) = \sum_{i=1}^n d(\alpha_1(i), \alpha_2(i)).$$

Finally, for elements $f_1, f_2 \in \coprod_A G_\lambda$ the Graev metric on the free product with amalgamation $\coprod_A G_\lambda$ is defined as

$$\underline{d}(f) = \inf \{ \rho(\alpha_1, \alpha_2) : |\alpha_1| = |\alpha_2| \text{ and } \hat{\alpha}_i = f_i \}.$$

Lemma 5.1. *\underline{d} is a tsi pseudo-metric.*

Proof. It is obvious that \underline{d} is non-negative, symmetric and attains value zero on the diagonal. We show that it is two-sided invariant. Let $f_1, f_2, h \in \coprod_A G_\lambda$ be given. Let $\gamma \in \text{Words}(G)$ be any word such that $\hat{\gamma} = h$. For any $\alpha_1, \alpha_2 \in \text{Words}(G)$ that have the same length and are such that $\hat{\alpha}_i = f_i$ we get

$$\rho(\alpha_1, \alpha_2) = \rho(\gamma \frown \alpha_1, \gamma \frown \alpha_2),$$

and therefore $\underline{d}(hf_1, hf_2) \leq \underline{d}(f_1, f_2)$. But similarly, if β_1, β_2 are of the same length and $\hat{\beta}_i = hf_i$, then

$$\rho(\beta_1, \beta_2) = \rho(\gamma^{-1} \frown \beta_1, \gamma^{-1} \frown \beta_2),$$

where $\gamma^{-1} = \gamma(|\gamma|)^{-1} \frown \dots \frown \gamma(1)^{-1}$. Hence $\underline{d}(f_1, f_2) = \underline{d}(hf_1, hf_2)$, i.e., \underline{d} is left-invariant. Right invariance is shown similarly.

We also need to check the triangle inequality. By the two-sided invariance triangle inequality is equivalent to

$$\underline{d}(f_1 f_2, e) \leq \underline{d}(f_1, e) + \underline{d}(f_2, e) \quad \text{for all } f_1, f_2 \in \coprod_A G_\lambda.$$

The latter follows immediately from the observation that if $\hat{\alpha}_i = f_i$, $|\alpha_i| = |\zeta_i|$, and $\hat{\zeta}_1 = e = \hat{\zeta}_2$, then $\widehat{\alpha_1 \frown \alpha_2} = f_1 f_2$, $\widehat{\zeta_1 \frown \zeta_2} = e$, and also

$$\rho(\alpha_1 \frown \alpha_2, \zeta_1 \frown \zeta_2) = \rho(\alpha_1, \zeta_1) + \rho(\alpha_2, \zeta_2). \quad \square$$

We will show eventually that, in fact, \underline{d} is not only a pseudo-metric, but a genuine metric. This will take us a while though.

It will be convenient for us to talk about norms rather than about metrics. For this we set $N(f) = \underline{d}(f, e)$. Then N is a tsi pseudo-norm on G (again, it will turn out to be a norm). Note that \underline{d} is a metric if and only if N is a norm, i. e., if and only if $N(f) = 0$ implies $f = e$.

5.2. Reductions. We start a series of reductions and will gradually simplify the structure of α in the definition of the pseudo-norm N .

Using the notion of an f -pair the definition of N can be rewritten as

$$N(f) = \inf \{ \rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is an } f\text{-pair} \}.$$

Lemma 5.2. *For all $f \in \coprod_A G_\lambda$*

$$N(f) = \inf \{ \rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is a congruent } f\text{-pair} \}.$$

Proof. Fix an $f \in \coprod_A G_\lambda$. We need to show that for any f -pair (α, ζ) and for any $\epsilon > 0$ there is a congruent f -pair (β, ξ) such that

$$\rho(\beta, \xi) \leq \rho(\alpha, \zeta) + \epsilon.$$

Take an f -pair (α, ζ) and fix an $\epsilon > 0$. Let n be the length of α . For an $i \in [1, n]$ we define a pair of words β_i, ξ_i as follows: if $\alpha(i) \cong \zeta(i)$, then $\beta_i = \alpha(i)$, $\xi_i = \zeta(i)$; if $\alpha(i) \not\cong \zeta(i)$, then $\beta_i = \alpha(i) \frown e$ and $\xi_i = a_i \frown a_i^{-1} \zeta(i)$, where $a_i \in A$ is any element such that

$$d(\alpha(i), \zeta(i)) + \frac{\epsilon}{n} \geq d(\alpha(i), a_i) + d(a_i, \zeta(i)),$$

which exists by the definition of the amalgam metric d . Then

$$\rho(\beta_i, \xi_i) \leq \rho(\alpha(i), \zeta(i)) + \frac{\epsilon}{n} \quad \text{for all } i.$$

Set $\beta = \beta_1 \frown \dots \frown \beta_n$, $\xi = \xi_1 \frown \dots \frown \xi_n$. It is now easy to see that (β, ξ) is a congruent f -pair and that indeed

$$\rho(\beta, \xi) \leq \rho(\alpha, \zeta) + \epsilon. \quad \square$$

The next lemma follows immediately from the two-sided invariance of the metrics d_λ .

Lemma 5.3. *Let (α, ζ) be a congruent pair of length n , and let $\{I_k\}_{k=1}^m$ be a right [left] transfer admissible sequence of intervals. If (β, ξ) is the right [left] $\{I_k\}_{k=1}^m$ -transfer of the pair (α, ζ) , then*

$$\rho(\alpha, \zeta) = \rho(\beta, \xi).$$

Lemma 5.4. *Let (α, ζ) be a congruent f -pair, and let T_ζ be an evaluation tree for ζ . There is a slim f -pair (β, ξ) such that*

- (i) $|\alpha| = |\beta|$;
- (ii) $\rho(\alpha, \zeta) = \rho(\beta, \xi)$;
- (iii) T_ζ is a slim evaluation tree for ξ ;
- (iv) if T_ζ is a balanced evaluation tree for ζ , then it is also balanced as an evaluation tree for ξ .

Proof. Let (α, ζ) be a congruent f -pair, let T_ζ be an evaluation tree for ζ , and let H_{T_ζ} denote the height of the tree T_ζ . We do an inductive construction of words (β_k, ξ_k) for $k = 0, \dots, H_{T_\zeta}$ and claim that $(\beta_{H_{T_\zeta}}, \xi_{H_{T_\zeta}})$ is as desired. We start by setting $(\beta_0, \xi_0) = (\alpha, \zeta)$.

Suppose the pair (β_k, ξ_k) has been constructed. Let $t_1, \dots, t_m \in T$ be all the nodes at the level $H_{T_\zeta} - k$ listed in the increasing order: $M(I_{t_i}) < M(I_{t_{i+1}})$. We define a relation \sim on $[1, m]$ by setting $k \sim l$ if for any $i \in [m(I_{t_k} \cup I_{t_l}), M(I_{t_k} \cup I_{t_l})]$ there is $j \in [1, m]$ such that $i \in I_{t_j}$. It is straightforward to check that \sim is an equivalence relation on $[1, m]$. Note that any \sim -equivalence class is a sub-interval of $[1, m]$. Let J_1, \dots, J_p be the increasing list of all the distinct equivalence classes, $J_1 < J_2 < \dots < J_p$.

Case 1. $p \geq 2$. Set (γ, ω) to be the right $\{I_{t_r}\}_{r=1}^{M(J_{p-1})}$ -transfer of (β_k, ξ_k) , and define (β_{k+1}, ξ_{k+1}) to be the left $\{I_{t_r}\}_{r=m(J_p)}^m$ -transfer of (γ, ω) .

Case 2. $p = 1$. Suppose there is only one equivalence class. We have a trichotomy:

- if $M(I_{M(J_1)}) < n$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the right } \{I_{t_r}\}_{r=1}^m\text{-transfer of } (\beta_k, \xi_k);$$

- if $M(I_{M(J_1)}) = n$, but $m(I_{m(J_1)}) > 1$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the left } \{I_{t_r}\}_{r=1}^m\text{-transfer of } (\beta_k, \xi_k);$$

- if $m(I_{m(J_1)}) = 1$ and $M(I_{M(J_1)}) = n$, then set

$$(\beta_{k+1}, \xi_{k+1}) = \text{the right } \{I_{t_r}\}_{r=1}^{m-1}\text{-transfer of } (\beta_k, \xi_k).$$

Notice the difference from the first case: the last element of the transfer sequence is $r = m - 1$, not m .

Denote $(\beta_{H_{T_\zeta}}, \xi_{H_{T_\zeta}})$ simply by (β, ξ) . We claim that this pair satisfies all the requirements. Since (β, ξ) is obtained by the sequence of transfers, items (i) and (iv) follow from Lemma 2.14. Item (ii) is a consequence of Lemma 5.3.

It remains to check that $\hat{\xi}[I_t] = e$ for all $t \in T_\zeta$. By item (v) of Lemma 2.14 $\hat{\xi}_{k+1}[I_t] = e$ for all $t \in T_\zeta$ such that $H_{T_\zeta}(t) = H_{T_\zeta} - k$. Therefore it is enough to show that $\hat{\xi}_{k+1}[I_t] = \hat{\xi}_k[I_t]$ for all $t \in T_\zeta$ such that $H_{T_\zeta}(t) > H_{T_\zeta} - k$. This follows

from item (iv) of Lemma 2.14 and item (v) of the definition of the evaluation tree. \square

Lemma 5.5. *Let (α, ζ) be a slim f -pair, and let T_ζ be a slim balanced evaluation tree for ζ . There is a simple f -pair (β, ξ) such that*

- (i) $|\alpha| = |\beta|$;
- (ii) $\rho(\alpha, \zeta) = \rho(\beta, \xi)$;
- (iii) T_ζ is a slim balanced evaluation tree for ξ .

Proof. Let (α, ζ) be a slim f -pair of length n , and let T_ζ be a slim evaluation tree for ζ . Sets $\{R_t\}_{t \in T_\zeta}$ form a partition of $[1, n]$. For $t \in T$ let $J_1^t, \dots, J_{q_t}^t$ be the maximal sub-intervals of R_t . Let $\{i_k\}_{k=1}^m$ be the list of external letters in ζ . Set

$$F(J_i^t) = \{i_k\} \cap J_i^t.$$

Assume first that $F(J_i^t) \neq \emptyset$ for all $t \in T_\zeta$ and all $i \in [1, q_t]$. Note that by item (vii) of the definition of the balanced evaluation tree this is the case once $T \neq \{\emptyset\}$. Set

$$U = \left(\bigcup_{t \in T_\zeta} \bigcup_{i=1}^{q_t} [m(J_i^t), M(F(J_i^t))] \right) \setminus \{i_k\}_{k=1}^m,$$

$$V = \left(\bigcup_{t \in T_\zeta} \bigcup_{i=1}^{q_t} [M(F(J_i^t)), M(J_i^t)] \right) \setminus \{i_k\}_{k=1}^m.$$

Now write $U = \{u_k\}_{k=1}^{p_u}$, $V = \{v_k\}_{k=1}^{p_v}$ as increasing sequences. Set (γ, ω) to be the right $\{u_k\}$ -transfer of the pair (α, ζ) and (β, ξ) to be the left $\{v_k\}$ -transfer of (γ, ω) (we view u_k 's and v_k 's as intervals that consist of a single point). We claim that the pair (β, ξ) satisfies all the assumptions of the lemma.

Item (i) follows from item (i) of Lemma 2.14. The latter lemma also implies that T_ζ is a balanced evaluation tree for ξ . Item (ii) follows from Lemma 5.3.

(iii). We show that T_ζ is a slim evaluation tree for ξ . Let $t \in T_\zeta$. Since T_ζ was slim for ζ , we have $\hat{\zeta}[I_t] = e$. Note that if $u_k \in U \cap R_t$, then $u_k + 1 \in R_t$ (by the construction of U). Similarly for $v_k \in V$, $v_k \in R_t$ implies $v_k - 1 \in R_t$. It now follows from item (iv) of Lemma 2.14 that $\hat{\xi}[I_t] = \hat{\zeta}[I_t] = e$ and therefore T_ζ is slim.

Finally, the simplicity of (β, ξ) is a consequence of items (iv) and (v) of Lemma 2.14.

So have we proved the lemma under the assumption that $F(J_i^t) \neq \emptyset$ for all $t \in T_\zeta$ and all $i \in [1, q_t]$. Suppose this assumption was false. By item (vii) of the definition of the balanced evaluation tree we get $T_\zeta = \{\emptyset\}$ and $F(I_\emptyset) = \emptyset$. Therefore $\zeta(i) \in A$ for all i . Set (β, ξ) to be the right $(i)_{i=1}^{n-1}$ -transfer of (α, ζ) . Then $\xi = e \cap \dots \cap e$ and obviously (β, ξ) is a simple f -pair of the same length and $T_\zeta = \{\emptyset\}$ is a simple balanced evaluation tree for ξ . \square

Lemma 5.6. *Let (α, ζ) be a slim f -pair of length n with a slim evaluation tree T_ζ . Let $t \in T_\zeta$ be given and let $\{i_k\}_{k=1}^m \subseteq R_t$ be a symmetrization admissible list. If $\xi = \text{Sym}(\alpha, \zeta; i', \{i_k\})$ for some $i' \in \{i_k\}_{k=1}^m$, then*

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi).$$

Proof. Since ζ is slim, we have

$$\zeta(i_1) \cdot \zeta(i_2) \cdots \zeta(i_m) = e,$$

and by Proposition 3.3 we get

$$d(\alpha(i_1) \cdots \alpha(i_m), e) = d(\alpha(i_1) \cdots \alpha(i_m), \zeta(i_1) \cdots \zeta(i_m)) \leq \sum_{j=1}^m d(\alpha(i_j), \zeta(i_j)).$$

If $i' = i_k$, then

$$\begin{aligned} \rho(\alpha, \zeta) - \rho(\alpha, \xi) &= \\ \sum_{j=1}^m d(\alpha(i_j), \zeta(i_j)) - d(\alpha(i_k), \alpha(i_{k-1})^{-1} \cdot \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \cdots \alpha(i_{k+1})^{-1}) &= \\ \sum_{j=1}^m d(\alpha(i_j), \zeta(i_j)) - d(\alpha(i_1) \cdots \alpha(i_m), e) &\geq 0. \end{aligned}$$

This proves the lemma. \square

Definition 5.7. A simple f -pair (α, ζ) is called *simple reduced* if α is a reduced form of f .

Lemma 5.8. For any $f \in \coprod_A G_\lambda$

$$N(f) = \inf\{\rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is a simple reduced } f\text{-pair}\}.$$

Proof. In view of Lemmas 5.2, 5.4, and 5.5, it is enough to show that for any simple f -pair (α, ζ) there is a simple reduced f -pair (β, ξ) such that $\rho(\alpha, \zeta) \geq \rho(\beta, \xi)$. Let (α, ζ) be a simple f -pair. Let (γ, ω) be a simple f -pair of the smallest length among all simple f -pairs (γ_0, ω_0) such that

$$\rho(\alpha, \zeta) \geq \rho(\gamma_0, \omega_0).$$

It is enough to show that γ is a reduced form of f . If $|\gamma| = 1$ this is obvious. Suppose $|\gamma| = n \geq 2$.

Claim 1. There is no $j \in [1, n]$ such that $\gamma(j) \in A$. Suppose this is false and there is such a $j \in [1, n]$.

Case 1. $\omega(j) \in A$. (In fact, since (γ, ω) is simple, $\omega(j) \in A$ implies $\omega(j) = e$, but this is not used here.) Suppose $j < n$. Since $\gamma(j) \in A$, $\omega(j) \in A$ and $\gamma(j+1) \cong \omega(j+1)$, we have $\gamma(j) \cdot \gamma(j+1) \cong \omega(j) \cdot \omega(j+1)$. Define (γ_1, ω_1) by

$$\begin{aligned} \gamma_1(i) &= \begin{cases} \gamma(i) & \text{if } i < j; \\ \gamma(j) \cdot \gamma(j+1) & \text{if } i = j; \\ \gamma(i+1) & \text{if } i > j; \end{cases} \\ \omega_1(i) &= \begin{cases} \omega(i) & \text{if } i < j; \\ \omega(j) \cdot \omega(j+1) & \text{if } i = j; \\ \omega(i+1) & \text{if } i > j. \end{cases} \end{aligned}$$

It is easy to see that $|\gamma_1| = |\gamma| - 1$ and (γ_1, ω_1) is a congruent f -pair. Moreover, since by the two-sided invariance

$$d(\gamma(j)\gamma(j+1), \omega(j)\omega(j+1)) \leq d(\gamma(j), \omega(j)) + d(\gamma(j+1), \omega(j+1)),$$

we also have $\rho(\gamma, \omega) \geq \rho(\gamma_1, \omega_1)$. Since γ_1, ω_1 is a congruent f -pair, by Lemmas 5.4 and 5.5 there is a simple f -pair (γ_0, ω_0) such that $|\gamma_0| = |\gamma_1| = n - 1$ and $\rho(\gamma_0, \omega_0) = \rho(\gamma_1, \omega_1)$. This contradicts the choice of (γ, ω) .

If $j = n$, define

$$\gamma_1(i) = \begin{cases} \gamma(i) & \text{if } i < j-1; \\ \gamma(j-1) \cdot \gamma(j) & \text{if } i = j-1; \\ \gamma(i+1) & \text{if } i > j-1; \end{cases}$$

$$\omega_1(i) = \begin{cases} \omega(i) & \text{if } i < j-1; \\ \omega(j-1) \cdot \omega(j) & \text{if } i = j-1; \\ \omega(i+1) & \text{if } i > j-1, \end{cases}$$

and proceed as before.

Case 2. $\omega(j) \notin A$. Let T_ω be a slim evaluation tree for ω . Let $t \in T_\omega$ be such that $j \in R_t$. Let $\{i_k\}_{k=1}^m$ be the list of external letters in R_t ; this list is symmetrization admissible. Let $j_0 \in \{i_k\}_{k=1}^m$ be any such that $j_0 \neq j$, set $\omega_2 = \text{Sym}(\gamma, \omega; j_0, \{i_k\})$. By Lemma 2.16 (γ, ω_2) is a slim f -pair and $\omega_2(j) = \gamma(j) \in A$. And we can decrease the length of the pair (γ, ω_2) as in the previous case. This proves the case and the claim.

Claim 2. There is no $j \in [1, n-1]$ such that $\gamma(j) \cong \gamma(j+1)$. Suppose this is false and there is such a $j \in [1, n-1]$. Note that by the previous claim $\gamma(j) \notin A$ and $\gamma(j+1) \notin A$. Hence there is $\lambda_0 \in \Lambda$ such that

$$\gamma(j), \gamma(j+1), \omega(j), \omega(j+1) \in G_{\lambda_0}.$$

Therefore $\gamma(j) \cdot \gamma(j+1) \cong \omega(j) \cdot \omega(j+1)$. The rest of the proof is similar to what we have done in the previous claim. Define (γ_3, ω_3) by

$$\gamma_3(i) = \begin{cases} \gamma(i) & \text{if } i < j \\ \gamma(j) \cdot \gamma(j+1) & \text{if } i = j \\ \gamma(i+1) & \text{if } i > j \end{cases}$$

$$\omega_3(i) = \begin{cases} \omega(i) & \text{if } i < j \\ \omega(j) \cdot \omega(j+1) & \text{if } i = j \\ \omega(i+1) & \text{if } i > j \end{cases}$$

Then $|\gamma_3| = |\gamma| - 1$, (γ_3, ω_3) is a congruent f -pair, and $\rho(\gamma, \omega) \geq \rho(\gamma_1, \omega_1)$. By Lemmas 5.4 and 5.5 there is a simple f -pair (γ_0, ω_0) such that $|\gamma_0| = |\gamma_3|$ and $\rho(\gamma_3, \omega_3) = \rho(\gamma_0, \omega_0)$, contradicting the choice of (γ, ω) . The claim is proved.

From the second claim it follows that $\gamma(j) \not\cong \gamma(j+1)$ for any $j \in [1, n-1]$ and therefore γ is reduced. \square

Proposition 5.9. *Let $f \in \coprod_A G_\lambda$ be an element of length n . If α is a reduced form of f , then*

$$N(f) \geq \min\{d(\alpha(i), A) : i \in [1, n]\}.$$

Proof. Fix a reduced form α of f , the word α has length n . By Lemma 5.8 it remains to show that for any simple reduced f -pair (β, ξ) we have

$$\rho(\beta, \xi) \geq \min\{d(\alpha(i), A) : i \in [1, n]\}.$$

Let (β, ξ) be a simple reduced f -pair. Note that by Lemma 2.4 the length of β is n . Let T_ξ be a slim evaluation tree for ξ , and let $t \in T_\xi$ be a leaf (i.e., a node with no predecessors). Since I_t is ξ -congruent and (β, ξ) is a simple reduced pair,

it follows that there is $i_0 \in I_t$ such that $\xi(i_0) = e$ (in fact, either $\xi(m(I_t)) = e$ or $\xi(m(I_t) + 1) = e$). By Lemma 2.4 there are $a_1, a_2 \in A$ such that $a_1 \alpha(i_0) a_2 = \beta(i_0)$. By the two-sided invariance we get

$$\rho(\beta, \xi) \geq d(\beta(i_0), e) = d(a_1 \alpha(i_0) a_2, e) = d(\alpha(i_0), a_1^{-1} a_2^{-1}) \geq d(\alpha(i_0), A). \quad \square$$

We are now ready to prove that the pseudo-metric \underline{d} is, in fact, a metric.

Theorem 5.10. *If \underline{d} is (as before) the pseudo-metric on $\coprod_A G_\lambda$ associated with the pseudo-norm N , $\underline{d}(f, e) = N(f)$, then*

- (i) \underline{d} is a two-sided invariant metric on $\coprod_A G_\lambda$;
- (ii) \underline{d} extends d .

Proof. (i) By Proposition 5.1 we know that \underline{d} is a tsi pseudo-metric. It only remains to show that $\underline{d}(f, e) = 0$ implies $f = e$. Let $f \in \coprod_A G_\lambda$ be such that $\underline{d}(f, e) = 0$, and let α be a reduced form of f . Suppose first that $|\alpha| \geq 2$ and therefore $\alpha(i) \notin A$ for all i by the definition of the reduced form. By Proposition 5.9 and since A is closed in G_λ for all λ , we have

$$\underline{d}(f, e) \geq \min \{d(\alpha(i), A) : i \in [1, |\alpha|]\} > 0.$$

Suppose now $|\alpha| = 1$ and therefore $\alpha = f$, $f \in G$, and the reduced form of f is unique. By Lemma 5.8 the distance $d(f, e)$ is given as the infimum over all simple reduced f -pairs, but there is only one such pair: (f, e) , where f is viewed as a letter in G . Hence $d(f, e) = 0$ implies $f = e$.

(ii) Fix $g_1, g_2 \in G$ and suppose first that $g_1 \not\cong g_2$. Let (α, ζ) be a simple reduced $g_1 g_2^{-1}$ -pair. We claim that there is $a \in A$ such that $g_1 a = \alpha(1)$, and $a^{-1} g_2^{-1} = \alpha(2)$. Indeed,

$$\begin{aligned} \alpha(1) \alpha(2) = g_1 g_2^{-1} &\implies g_2 g_1^{-1} \alpha(1) \alpha(2) = e \implies g_1^{-1} \alpha(1) \in A \implies \\ &\exists a \in A \text{ such that } \alpha(1) = g_1 a, \text{ and } \alpha(2) = a^{-1} g_2^{-1}. \end{aligned}$$

Moreover, since $g_1 \not\cong g_2$ and since (α, ζ) is congruent, we get $\zeta = e \frown e$ and thus

$$\begin{aligned} \underline{d}(g_1, g_2) &= \underline{d}(g_1 g_2^{-1}, e) = \inf \{ \rho(g_1 a \frown a^{-1} g_2^{-1}, e \frown e) : a \in A \} = \\ &= \inf \{ d(g_1, a^{-1}) + d(a^{-1}, g_2) : a \in A \} = d(g_1, g_2). \end{aligned}$$

If $g_1 \cong g_2$, then there is only one simple reduced $g_1 g_2^{-1}$ -pair, namely $(g_1 g^{-1}, e)$ and the item follows. \square

6. PROPERTIES OF GRAEV METRICS

Theorem 5.10 allows us to make the following definition: the metric \underline{d} constructed in the previous section is called the *Graev metric* on the free product of groups (G_λ, d_λ) with amalgamation over A .

Theorem 4.3 implies that the Graev metric on a free group is, in some sense, computable, that is if one can compute the metric on the base, then to find the norm of an element f in the free group one has to calculate the function ρ for only *finitely many* trivial words, moreover those words are constructable from the letters of f . For the case of free products without amalgamation, i.e., when $A = \{e\}$, we have a similar result (see Corollary 6.4 below).

Definition 6.1. Let (α, ζ) be a slim f -pair with a slim evaluation tree T_ζ . The pair (α, ζ) is called *symmetric with respect to the tree T_ζ* if for each $t \in T_\zeta$ there are a symmetrization admissible list $\{i_{t,k}\}_{k=1}^{m_t}$ and $j_t \in \{i_{t,k}\}_{k=1}^{m_t}$ such that

$$\zeta = \text{Sym}(\alpha, \zeta; j_t, \{i_{t,k}\}_{k=1}^{m_t}).$$

An f -pair (α, ζ) is called *symmetric* if there is a slim evaluation tree T_ζ such that (α, ζ) is a symmetric f -pair with respect to T_ζ .

Remark 6.2. Note that for any word α there are only finitely many words ζ such that (α, ζ) is symmetric.

Proposition 6.3. If $f \in \coprod_A G_\lambda$, then

$$N(f) = \inf\{\rho(\alpha, \xi) : (\alpha, \xi) \text{ is a symmetric reduced } f\text{-pair}\}.$$

Proof. By Lemma 5.8 it is enough to show that for any simple reduced f -pair (α, ζ) there is a symmetric reduced f -pair (α, ξ) such that

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi).$$

Let (α, ζ) be a simple reduced f -pair, and let T_ζ be a slim evaluation tree for ζ . We construct a new slim evaluation tree T_ζ^* for ζ with the following property: for any $t \in T_\zeta^*$ and any $i \in R_\zeta^*$ if $\zeta(i) = e$, then t is a leaf and, moreover, $R_t^* = I_t^* = \{i\}$.

Let $\{j_k\}_{k=1}^m$ be such that $\zeta(j_k) = e$ for all k and $\zeta(j) = e$ implies $j = j_k$ for some $k \in [1, m]$. We construct a sequence of slim evaluation trees $T_\zeta^{(k)}$ for ζ and claim that $T_\zeta^{(m)}$ is as desired. Set $T_\zeta^{(0)} = T_\zeta$. Suppose $T_\zeta^{(k)}$ has been constructed. Let $t_0 \in T_\zeta^{(k)}$ be such that $j_{k+1} \in R_{t_0}^{(k)}$. If $|R_{t_0}^{(k)}| = 1$, that is if $R_{t_0}^{(k)} = I^{(k)} = \{j_{k+1}\}$, then do nothing: set $T_\zeta^{(k+1)} = T_\zeta^{(k)}$.

Suppose $|R_{t_0}^{(k)}| > 1$. Let s be a symbol for a new node. For all $t \in T_\zeta^{(k)} \setminus \{t_0\}$ set

$$T_\zeta^{(k+1)} = T_\zeta^{(k)} \cup \{s\}, \quad I_t^{(k+1)} = I_t^{(k)}, \quad I_s^{(k+1)} = [j_{k+1}, j_{k+1}] = \{j_{k+1}\}.$$

We need to turn the set $T_\zeta^{(k+1)}$ into a tree. For that let the ordering of the nodes in $T_\zeta^{(k+1)}$ extend the ordering of the nodes of $T_\zeta^{(k)}$. To finish the construction it remains to define the place for the node s inside $T_\zeta^{(k+1)}$ and an interval $I_{t_0}^{(k+1)}$.

- If j_{k+1} is the minimal element of $R_{t_0}^{(k)}$, i.e., if $j_{k+1} = m(R_{t_0}^{(k)})$, then set $I_{t_0}^{(k+1)} = [m(I_{t_0}^{(k)}) + 1, M(I_{t_0}^{(k)})]$. Let $t_1 \in T_\zeta^{(k)}$ be such that $(t_0, t_1) \in E(T_\zeta^{(k)})$. Set $(s, t_1) \in E(T_\zeta^{(k+1)})$, or in other words, $s \prec t_1$ in $T_\zeta^{(k+1)}$.
- If j_{k+1} is the maximal element of $R_{t_0}^{(k)}$, i.e., if $j_{k+1} = M(R_{t_0}^{(k)})$, then set $I_{t_0}^{(k+1)} = [m(I_{t_0}^{(k)}), M(I_{t_0}^{(k)}) - 1]$. Let $t_1 \in T_\zeta^{(k)}$ be such that $(t_0, t_1) \in E(T_\zeta^{(k)})$. Set $(s, t_1) \in E(T_\zeta^{(k+1)})$, or in other words, $s \prec t_1$ in $T_\zeta^{(k+1)}$.
- If j_{k+1} is neither maximal nor minimal element of $R_{t_0}^{(k)}$, then set $I_{t_0}^{(k+1)} = I_{t_0}^{(k)}$ and $(s, t_0) \in E(T_\zeta^{(k+1)})$.

It is straightforward to check that $T_\zeta^{(k+1)}$ is a slim evaluation tree for ζ .

Finally, we define $T_\zeta^* = T_\zeta^m$. Then T_ζ^* is a slim evaluation tree for ζ and, by construction, if j is such that $\zeta(j) = e$, then $I_{t_0}^* = \{j\}$ for some $t_0 \in T_\zeta^*$.

Let $\{i_k\}_{k=1}^p$ be the list of external letters of ζ . Set

$$F_t^* = \begin{cases} R_t^* \cap \{i_k\}_{k=1}^p & \text{if } R_t^* \cap \{i_k\}_{k=1}^p \neq \emptyset; \\ I_t^* & \text{otherwise.} \end{cases}$$

Note that F_t^* is symmetrization admissible for all t . Let $\{t_j\}_{j=1}^N$ be the list of nodes of T_ζ^* . For any $j \in [1, N]$ pick some l_j such that $l_j \in F_{t_j}^*$. Set $\xi_0 = \zeta$ and construct inductively

$$\xi_{k+1} = \text{Sym}(\alpha, \xi_k; l_{k+1}, F_{t_{k+1}}^*).$$

Finally, set $\xi = \xi_N$. It follows from Lemma 2.16 that (α, ξ) is a slim f -pair and is symmetric with respect to T_ζ^* by construction. Lemma 5.6 implies

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi)$$

as desired. \square

If $A = \{e\}$, that is we have a free product without amalgamation, then for any $f \in \coprod_A G_\lambda$ there is exactly one reduced word $\alpha \in \text{Words}(G)$ such that $\hat{\alpha} = f$. This observation together with Remark 6.2 gives us the following

Corollary 6.4. *If $A = \{e\}$, then for any $f \in \coprod_A G_\lambda$*

$$N(f) = \min\{\rho(\alpha, \xi) : (\alpha, \xi) \text{ is a symmetric reduced } f\text{-pair}\}.$$

We can now prove an analog of Proposition 4.4 for the Graev metrics on the free products with amalgamation.

Proposition 6.5. *The Graev metric \underline{d} has the following properties:*

- (i) *if (T, d_T) is a tsi group, $\phi_\lambda : G_\lambda \rightarrow T$ are K -Lipschitz homomorphisms (K does not depend on λ) such that for all $a \in A$ and all $\lambda_1, \lambda_2 \in \Lambda$*

$$\phi_{\lambda_1}(a) = \phi_{\lambda_2}(a),$$

then there exist a unique K -Lipschitz homomorphism $\phi : \coprod_A G_\lambda \rightarrow T$ that extends ϕ_λ ;

- (ii) *let $H_\lambda < G_\lambda$ be subgroups such that $A < H_\lambda$ for all λ and think of $\coprod_A H_\lambda$ as being a subgroup of $\coprod_A G_\lambda$. Endow H_λ with the metric induced from G_λ . The Graev metric on $\coprod_A H_\lambda$ is the same as the induced Graev metric from $\coprod_A G_\lambda$. Moreover, if H_λ are closed subgroups, then $\coprod_A H_\lambda$ is a closed subgroup $\coprod_A G_\lambda$;*
- (iii) *let δ be any other tsi metric on the amalgam $\coprod_A G_\lambda$. If δ extends d , then $\delta(f_1, f_2) \leq \underline{d}(f_1, f_2)$ for all $f_1, f_2 \in \coprod_A G_\lambda$, i.e., \underline{d} is maximal among all the tsi metrics that extend d ;*
- (iv) *if $\Lambda' = \{\lambda \in \Lambda : G_\lambda \neq A\}$ and $|\Lambda'| \geq 2$, then*

$$\chi(\coprod_A G_\lambda) = \max\{\aleph_0, \sup\{\chi(G_\lambda) : \lambda \in \Lambda'\}, |\Lambda'|\}.$$

In particular, if Λ is at most countable and G_λ are all separable, then the amalgam is also separable.

Proof. (i) By the universal property for the free products with amalgamation there is a unique extension of the homomorphisms ϕ_λ to a homomorphism $\phi : \coprod_A G_\lambda \rightarrow$

T , it remains to check that ϕ is K -Lipschitz. Let (α, ζ) be a congruent f -pair of length n . Then

$$\begin{aligned} K\rho(\alpha, \zeta) &= \sum_{i=1}^n Kd(\alpha(i), \zeta(i)) \geq \sum_{i=1}^n d_T(\phi(\alpha(i)), \phi(\zeta(i))) \geq \\ &= d_T(\phi(\hat{\alpha}), \phi(\hat{\zeta})) = d_T(\phi(f), e). \end{aligned}$$

And therefore

$$K\underline{d}(f, e) = \inf\{K\rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is a congruent } f\text{-pair}\} \geq d_T(\phi(f), e).$$

Hence ϕ is K -Lipschitz.

(ii) Let \underline{d}_H be the Graev metric on $\coprod_A H_\lambda$ and \underline{d} be the Graev metric on $\coprod_A G_\lambda$. From Proposition 6.3 it follows that $\underline{d}_H = \underline{d}|_{\coprod_A H_\lambda}$.

For the moreover part suppose that H_λ are closed in G_λ for all $\lambda \in \Lambda$. Set $H = \bigcup_{\lambda \in \Lambda} H_\lambda$. Note that H is a closed subset of G . Suppose towards a contradiction that there exists $f \in \coprod_A G_\lambda$ such that $f \notin \coprod_A H_\lambda$, but $f \in \overline{\coprod_A H_\lambda}$. Let $\alpha \in \text{Words}(G)$ be a reduced form of f , and let $n = |\alpha|$. Set

$$\begin{aligned} \epsilon_1 &= \min\{d(\alpha(i), A) : i \in [1, n]\}, \\ \epsilon_2 &= \min\{d(\alpha(i), H) : i \in [1, n], \alpha(i) \notin H\}. \end{aligned}$$

Note that $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Let $i_0 \in [1, n]$ be the largest such that $\alpha(i_0) \notin H$. By Lemma 2.4 the numbers ϵ_i and i_0 are independent of the choice of the reduced form α . Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Let $h \in \coprod_A H_\lambda$ be such that $\underline{d}(f, h) < \epsilon$. By Lemma 5.8 there is a simple reduced fh^{-1} -pair (β, ξ) such that $\rho(\beta, \xi) < \epsilon$. Let T_ξ be a slim evaluation tree for ξ , and let $t_0 \in T_\xi$ be such that $i_0 \in R_{t_0}$. It is easy to see that there is a word α' such that α' is a reduced form of f , $\alpha'(i) = \beta(i)$ for all $i \in [1, i_0 - 1]$, and $\alpha'(i_0) = \beta(i_0) \cdot h_0$ for some $h_0 \in H$. Without loss of generality assume that $\alpha' = \alpha$. Note that $\beta(i) \in H$ for all $i > i_0$.

We claim that $i_0 = m(R_{t_0})$. Suppose not. Let $j_0 \in R_{t_0}$ be such that $j_0 < i_0$ and $[j_0 + 1, i_0 - 1] \cap R_{t_0} = \emptyset$ (i.e., j_0 is the predecessor of i_0 in R_{t_0}). Let $I = [j_0 + 1, i_0 - 1]$. Because T_ξ is slim, $\hat{\xi}[I] = e$. Since β is reduced and (β, ξ) is congruent, there is $i_1 \in I$ such that $\xi(i_1) \in A$ (in fact, $\xi(i_1) = e$). But then

$$\rho(\beta, \xi) \geq d(\beta(i_1), \xi(i_1)) \geq d(\alpha(i_1), A) \geq \epsilon,$$

contradicting the assumption $\rho(\beta, \xi) < \epsilon$. The claim is proved.

Therefore $i_0 = m(R_{t_0})$. Let $\{j_k\}_{k=1}^m$ be the list of external letters of ξ , and let $F_{t_0} = R_{t_0} \cap \{j_k\}_{k=1}^m$. We know that $\xi(i_0) \notin A$, since otherwise $\rho(\beta, \xi) \geq \epsilon$. Thus $i_0 \in F_{t_0}$. Let $\xi' = \text{Sym}(\beta, \xi; i_0, F_{t_0})$. By Lemma 5.6 $\rho(\beta, \xi) \geq \rho(\beta, \xi')$. Since $\beta(i) \in H$ for all $i > i_0$, we get $\xi'(i) \in H$ for all $i \in R_{t_0} \setminus \{i_0\}$. Let λ_0 be such that $\xi'(i) \in H_{\lambda_0}$ for all $i \in R_{t_0} \setminus \{i_0\}$. Since $\hat{\xi}'[R_{t_0}] = e$, it follows that $\xi'(i_0) \in H_{\lambda_0}$ as well. Finally, we get

$$\rho(\beta, \xi) \geq \rho(\beta, \xi') \geq d(\beta(i_0), \xi'(i_0)) \geq d(\alpha(i_0), H_{\lambda_0}) \geq \epsilon,$$

contradicting the choice of (β, ξ) . Therefore there is no $f \in \overline{\coprod_A H_\lambda}$ such that $f \notin \coprod_A H_\lambda$.

(iii) Let $f \in \coprod_A G_\lambda$ be given, let (α, ζ) be a congruent f -pair of length n . Since δ extends \underline{d} , we get

$$\delta(f, e) \leq \sum_{i=1}^n \delta(\alpha(i), \zeta(i)) = \sum_{i=1}^n \underline{d}(\alpha(i), \zeta(i)).$$

By taking the infimum over all such pairs (α, ζ) we get $\delta(f, e) \leq \underline{d}(f, e)$. By the left-invariance $\delta(f_1, f_2) \leq \underline{d}(f_1, f_2)$ for all $f_1, f_2 \in \coprod_A G_\lambda$.

(iv) If $|\Lambda'| \geq 2$, then $\coprod_A G_\lambda$ is an infinite metric space, therefore $\chi(\coprod_A G_\lambda) \geq \aleph_0$. Since $G_\lambda < \coprod_A G_\lambda$, it follows that $\chi(\coprod_A G_\lambda) \geq \chi(G_\lambda)$. We now show that $\chi(\coprod_A G_\lambda) \geq |\Lambda'|$. It is enough to consider the case $|\Lambda'| \geq \aleph_0$. There is an $\epsilon_0 > 0$ such that

$$|\{\lambda \in \Lambda : \sup\{d(g, A) : g \in G_\lambda\} > \epsilon_0\}| = |\Lambda'|.$$

For any such λ choose a $g_\lambda \in G_\lambda$ such that $d(g_\lambda, A) > \epsilon_0$. The family $\{g_\lambda\}_{\lambda \in \Lambda}$ is $2\epsilon_0$ -separated and hence $\chi(\coprod_A G_\lambda) \geq |\Lambda'|$.

Finally, for the reverse inequality, let $F_\lambda \subseteq G_\lambda$ be dense sets such that $|F_\lambda| = \chi(G_\lambda)$. The set

$$\left\{ \hat{\alpha} : \alpha \in \text{Words}\left(\bigcup_{\lambda \in \Lambda} F_\lambda\right) \right\}$$

is dense in $\coprod_A G_\lambda$ and

$$\left| \text{Words}\left(\bigcup_{\lambda \in \Lambda} F_\lambda\right) \right| = \max \left\{ \aleph_0, \sup\{\chi(G_\lambda) : \lambda \in \Lambda\}, |\Lambda'| \right\}. \quad \square$$

6.1. Factors of Graev metrics. Note that one can naturally view G as a pointed metric space (G, e, d) , and the identity map $G \mapsto \coprod_A G_\lambda$ is 1-Lipschitz (in fact, we have shown in Theorem 5.10 that it is an isometric embedding). We can construct the Graev metric on the free group $(F(G), d_F)$, and by item (i) of Proposition 4.4 there is a 1-Lipschitz homomorphism

$$\phi : F(G) \rightarrow \coprod_A G_\lambda$$

such that $\phi(g) = g$ for all $g \in G$. Since G generates $\coprod_A G_\lambda$, the map ϕ is onto. Let $\mathfrak{N} = \ker(\phi)$ be the kernel of this homomorphism. If d_0 is the factor metric on $F(G)/\mathfrak{N}$ (see the remark after Proposition 3.4), then $(F(G)/\mathfrak{N}, d_0)$ is a tsi group and $F(G)/\mathfrak{N}$ is isomorphic to $\coprod_A G_\lambda$ as an abstract group.

Proposition 6.6. *In the above setting $(F(G)/\mathfrak{N}, d_0)$ is isometrically isomorphic to $(\coprod_A G_\lambda, \underline{d})$.*

Proof. We recall the definition of the factor metric: for $f_1\mathfrak{N}, f_2\mathfrak{N} \in F(G)/\mathfrak{N}$

$$d_0(f_1\mathfrak{N}, f_2\mathfrak{N}) = \inf\{d_F(f_1h_1, f_2h_2) : h_1, h_2 \in \mathfrak{N}\}.$$

Of course, by construction $F(G)/\mathfrak{N}$ is isomorphic to $\coprod_A G_\lambda$ and we check that the natural isomorphism is an isometry.

Let $f' \in \coprod_A G_\lambda$, and let $w \in \text{Words}(G)$ be reduced form of f' . We can naturally view w as a reduced form of the element in $F(G)$, call it f . It is enough to show that for any such f and f' we have

$$d_0(f'\mathfrak{N}, \mathfrak{N}) = \underline{d}(f, e).$$

Note that if $h \in \mathfrak{N}$, then $h^\# \in \mathfrak{N}$ (for the definition of $h^\#$ see Subsection 4.1). Therefore by Proposition 4.5

$$d_0(f'\mathfrak{N}, \mathfrak{N}) = \inf\{d_F(f'h, e) : h \in \mathfrak{N}\} = \inf\{d_F(f'h^\#, e) : h \in \mathfrak{N}\}.$$

If $h \in \mathfrak{N}$ and $\gamma \in \text{Words}(G)$ is the reduced form of $h^\# \in F(G)$, then

$$d_F(f'h^\#, e) = \inf\left\{\rho(w \frown \gamma, (w \frown \gamma)^\theta) : \theta \text{ is a match on } [1, |w \frown \gamma|]\right\}.$$

Since $w, \gamma \in \text{Words}(G)$ and since $\hat{\gamma} = e$, we get $\underline{d}(f, e) \leq d_0(w\mathfrak{N}, \mathfrak{N})$. Since f was arbitrary and because of left-invariance of the metrics \underline{d} and d_0 , we get $\underline{d} \leq d_0$.

For the reverse inequality note that d_0 is a two-sided invariant metric on $\coprod_A G_\lambda$ and it extends the metric d on G , therefore by item (iii) of Proposition 6.5 we have $d_0 \leq \underline{d}$ and hence $d_0 = \underline{d}$. \square

6.2. Graev metrics for products of Polish groups. We would like to note that the construction of metrics on the free products with amalgamation works well with respect to group completions. Let us be more precise. Suppose we start with tsi groups (G_λ, d_λ) and a common closed subgroup $A < G_\lambda$, assume additionally that all the groups G_λ are complete as metrics spaces. The group $(\coprod_A G_\lambda, \underline{d})$, in general, is not complete, so let's take its group completion (for tsi groups this is the same as the metric completion), which we denote by $(\overline{\coprod_A G_\lambda}, \underline{d})$. We have an analog of item (i) of Proposition 6.5 for complete tsi groups. But first we need a simple lemma.

Lemma 6.7. *Let (H_1, d_1) and (H_2, d_2) be complete tsi groups, $\Lambda < H_1$ be a dense subgroup and $\phi : \Lambda \rightarrow H_2$ be a K -Lipschitz homomorphism. Then ϕ extends uniquely to a K -Lipschitz homomorphism*

$$\psi : H_1 \rightarrow H_2.$$

Proof. Let $h \in H_1$ and let $\{b_n\}_{n=1}^\infty \subseteq \Lambda$ be such that $b_n \rightarrow h$. Since ψ is K -Lipschitz, we have

$$d_2(\psi(b_n), \psi(b_m)) \leq K d_1(b_n, b_m).$$

Hence $\{\psi(b_n)\}_{n=1}^\infty$ is a d_2 -Cauchy sequence, and thus there is $f \in H_2$ such that $\psi(b_n) \rightarrow f$. Set $\psi(h) = f$. This extends ψ to a map $\psi : H_1 \rightarrow H_2$ and it is easy to see that its extension is still K -Lipschitz. \square

Combining the above result with item (i) of Proposition 6.5 we get

Proposition 6.8. *Let (T, d_T) be a complete tsi group, let $\phi_\lambda : G_\lambda \rightarrow T$ be K -Lipschitz homomorphisms such that for all $a \in A$ and all $\lambda_1, \lambda_2 \in \Lambda$*

$$\phi_{\lambda_1}(a) = \phi_{\lambda_2}(a).$$

There exist a unique K -Lipschitz homomorphism $\phi : \overline{\coprod_A G_\lambda} \rightarrow T$ such that ϕ extends ϕ_λ for all λ .

This proposition together with item (iv) of Proposition 6.5 shows that there are countable coproducts in the category of tsi Polish metric groups and 1-Lipschitz homomorphisms.

6.3. Tsi groups with no Lie sums and Lie brackets. In [11] L. van den Dries and S. Gao gave an example of a group, which they denote by F , and a two-sided invariant metric d on F such that the completion (\overline{F}, d) of this group has neither Lie sums nor Lie brackets. More precisely, they constructed two one-parameter subgroups

$$A_i = \left(f_t^{(i)} \right)_{t \in \mathbb{R}} < \overline{F} \quad i = 1, 2,$$

such that neither Lie sum nor Lie bracket of A_1 and A_2 exist.

Their group can be nicely explained in our setting. It turns out that the group F that they have constructed is isometrically isomorphic to the group $\mathbb{Q} * \mathbb{Q}$ with the Graev metric (and the metrics on the copies of the rationals are the usual absolute-value metrics). The group completion of $\mathbb{Q} * \mathbb{Q}$ is then the same as the group completion of the group $\mathbb{R} * \mathbb{R}$ with the Graev metric. And moreover, A_1 and A_2 are just the one-parameter subgroups given by the \mathbb{R} factors.

7. METRICS ON SIN GROUPS

Recall that topological group is SIN if for every open neighborhood of the identity there is a smaller open neighborhood $V \subseteq G$ such that $gVg^{-1} = V$ for all $g \in G$. SIN stands for Small Invariant Neighborhoods. It is well-known that a metrizable topological group admits a compatible two-sided invariant metric if and only if it is a SIN group.

Suppose G_λ are metrizable topological groups that admit compatible two-sided invariant metrics and $A < G_\lambda$ is a common closed subgroup. It is natural to ask whether one can find compatible tsi metrics d_λ that agree on A .

Question 7.1. *Let G_1 and G_2 be metrizable SIN topological groups, and let $A < G_i$ be a common closed subgroup. Are there compatible tsi metrics d_i on G_i such that*

$$d_1(a_1, a_2) = d_2(a_1, a_2)$$

for all $a_1, a_2 \in A$?

We do not know the answer to this question. Before discussing some partial results let us recall the notion of a Birkhoff-Kakutani family of neighborhoods.

Definition 7.2. Let G be a topological group. A family $\{U_i\}_{i=0}^\infty$ of open neighborhoods of the identity $e \in G$ is called *Birkhoff-Kakutani* if the following conditions are met:

- (i) $U_0 = G$;
- (ii) $\bigcap_i U_i = e$;
- (iii) $U_i^{-1} = U_i$;
- (iv) $U_{i+1}^3 \subseteq U_i$.

If additionally

- (v) $gU_i g^{-1} = U_i$ for all $g \in G$,

then the sequence is called *conjugacy invariant*.

It is well known (see, for example, [2]) that a topological group G admits a Birkhoff-Kakutani family if and only if it is metrizable. Moreover, let $\{U_i\}_{i=0}^\infty$ be a Birkhoff-Kakutani family in a group G , for $g_1, g_2 \in G$ set

$$\eta(g_1, g_2) = \inf\{2^{-n} : g_2^{-1}g_1 \in U_n\},$$

$$d(g_1, g_2) = \inf \left\{ \sum_{i=1}^{n-1} \eta(f_i, f_{i+1}) : \{f_i\}_{i=1}^n \subseteq G, f_1 = g_1, f_n = g_2 \right\}.$$

Then the function d is a compatible left-invariant metric on G and for all $g_1, g_2 \in G$

$$\frac{1}{2}\eta(g_1, g_2) \leq d(g_1, g_2) \leq \eta(g_1, g_2).$$

We call this metric d a *Birkhoff-Kakutani metric* associated with the family $\{U_i\}$.

A metrizable topological group admits a compatible tsi metric if and only if there is a conjugacy invariant Birkhoff-Kakutani family, and moreover, if $\{U_i\}$ is conjugacy invariant, then the metric d constructed above is two-sided invariant.

Proposition 7.3. *Let G_1 and G_2 be metrizable SIN groups, let $A < G_i$ be a common subgroup. There are compatible tsi metrics d_i on G_i such that $d_1|_A$ is bi-Lipschitz equivalent to $d_2|_A$, i.e., there is $K > 0$ such that*

$$\frac{1}{K}d_1(a_1, a_2) \leq d_2(a_1, a_2) \leq Kd_1(a_1, a_2)$$

for all $a_1, a_2 \in A$.

Proof. Since G_1 and G_2 are metrizable, we can fix two compatible metrics μ_1 and μ_2 on G_1 and G_2 respectively such that $\mu_i\text{-diam}(G_i) < 1$. We construct conjugacy invariant Birkhoff-Kakutani families $\{U_i^{(j)}\}_{i=0}^\infty$ for G_j , $j = 1, 2$, such that

- (i) $U_{2i+1}^{(1)} \cap A \subseteq U_{2i}^{(2)} \cap A$;
- (ii) $U_{2i+2}^{(2)} \cap A \subseteq U_{2i+1}^{(1)} \cap A$.

For the base of construction let $U_0^j = G_j$. Suppose we have constructed $\{U_i^{(j)}\}_{i=1}^N$ and suppose N is even (if N is odd, switch the roles of G_1 and G_2). If $V = U_N^{(2)} \cap A$, then V is an open neighborhood of the identity in A and therefore there is an open set $U \subseteq G_1$ such that $U \cap A = V$. Let $U_{N+1}^{(1)} \subseteq G_1$ be any open neighborhood of the identity such that $(U_{N+1}^{(1)})^{-1} = U_{N+1}^{(1)}$, $gU_{N+1}^{(1)}g^{-1} = U_{N+1}^{(1)}$ for all $g \in G_1$, $\mu_1\text{-diam}(U_{N+1}^{(1)}) < 1/N$ and

$$(U_{N+1}^{(1)})^3 \subseteq U \cap U_N^{(1)}.$$

Such a $U_{N+1}^{(1)}$ exists because G_1 is SIN. Set $U_{N+1}^{(2)}$ to be any open symmetric neighborhood of $e \in G_2$ such that $(U_{N+1}^{(2)})^3 \subseteq U_N^{(2)}$.

It is straightforward to check that such sequences $\{U_i^{(j)}\}_{i=1}^\infty$ indeed satisfy all the requirements. If d_j are the Birkhoff-Kakutani metrics that correspond to the families $\{U_i^{(j)}\}$, then for all $a_1, a_2 \in A$

$$\frac{1}{2}\eta_1(a_1, a_2) \leq \eta_2(a_1, a_2) \leq 2\eta_1(a_1, a_2),$$

whence

$$\frac{1}{4}d_1(a_1, a_2) \leq d_2(a_1, a_2) \leq 4d_1(a_1, a_2),$$

and therefore $d_1|_A$ and $d_2|_A$ are bi-Lipschitz equivalent with a constant $K = 4$. \square

Remark 7.4. It is, of course, straightforward to generalize the above construction to the case of finitely many groups G_j , but we do not know if the result is true for infinitely many groups G_j .

Remark 7.5. Note that one can always multiply the metric d_2 by a suitable constant (which is 4 in the above construction) to assure that $d_1|_A \leq d_2|_A$. We use this observation later in Remark 7.7.

Proposition 7.6. *Let G be a topological group, $A < G$ be a closed subgroup of G , N_G be a tsi norm on G , N_A be a tsi norm on A and suppose that for all $a \in A$*

$$N_A(a) \leq N_G(a).$$

There exists a compatible norm N on G such that

- (i) N extends N_A , that is $N_A(a) = N(a)$ for all $a \in A$;*
- (ii) $N \leq N_G$.*

If, moreover, A is a normal subgroup of G , then N is two-sided invariant.

Proof. For $g \in G$ set

$$N(g) = \inf\{N_A(a) + N_G(a^{-1}g) : a \in A\}.$$

We claim that N is a pseudo-norm on G .

- $N(e) = 0$ is obvious.
- For any $g \in G$ and any $a \in A$ by the two-sided invariance of N_G

$$N_A(a) + N_G(a^{-1}g) = N_A(a^{-1}) + N_G(g^{-1}a) = N_A(a^{-1}) + N_G(ag^{-1})$$

and therefore $N(g) = N(g^{-1})$.

- If $g_1, g_2 \in G$, then

$$\begin{aligned} N(g_1g_2) &= \inf\{N_A(a) + N_G(a^{-1}g_1g_2) : a \in A\} = \\ &= \inf\{N_A(a_1a_2) + N_G(a_2^{-1}a_1^{-1}g_1g_2) : a_1, a_2 \in A\} \leq \\ &= \inf\{N_A(a_1) + N_A(a_2) + N_G(a_1^{-1}g_1) + N_G(g_2a_2^{-1}) : a_1, a_2 \in A\} = \\ &= \inf\{N_A(a_1) + N_G(a_1^{-1}g_1) : a_1 \in A\} + \\ &= \inf\{N_A(a_2) + N_G(a_2^{-1}g_2) : a_2 \in A\} = \\ &= N(g_1) + N(g_2). \end{aligned}$$

Next we show that N is a compatible pseudo-norm. For a sequence $\{g_n\}_{n=1}^\infty \subseteq G$ we have

$$\begin{aligned} N(g_n) \rightarrow 0 &\iff \exists \{a_n\}_{n=1}^\infty \subseteq A \quad N_A(a_n) + N_G(a_n^{-1}g_n) \rightarrow 0 \iff \\ &\iff a_n \rightarrow e \text{ and } a_n^{-1}g_n \rightarrow e \iff \\ &\iff g_n \rightarrow e. \end{aligned}$$

In particular, N is a norm.

- (i) Now we claim that N extends N_A . Let $b \in A$. Using $N_G \geq N_A$ we get

$$\begin{aligned} N(b) &= \inf\{N_A(a) + N_G(a^{-1}b) : a \in A\} \geq \\ &\geq \inf\{N_A(a) + N_A(a^{-1}b) : a \in A\} \geq N_A(b). \end{aligned}$$

On the other hand

$$N(b) \leq N_A(b) + N_G(b^{-1}b) = N_A(b),$$

and therefore $N(b) = N_A(b)$.

(ii) Finally, for any $g \in G$ we have

$$\begin{aligned} N(g) &= \inf\{N_A(a) + N_G(a^{-1}g) : a \in A\} \leq \\ &\quad \inf\{N_G(a) + N_G(a^{-1}g) : a \in A\} \leq \\ &\quad N_G(e) + N_G(g) = N_G(g), \end{aligned}$$

and therefore $N \leq N_G$.

For the moreover part suppose that A is a normal subgroup. If $g_1 \in G$, then

$$\begin{aligned} N(g_1 g g_1^{-1}) &= \inf\{N_A(a) + N_G(a^{-1} g_1 g g_1^{-1}) : a \in A\} = \\ &\quad \inf\{N_A(g_1^{-1} a g_1) + N_G(g_1^{-1} a^{-1} g_1 g) : a \in A\} = N(g), \end{aligned}$$

and so N is two-sided invariant. \square

Remark 7.7. Proposition 7.3 (with Remark 7.5) and Proposition 7.6 together yield a positive answer to Question 7.1 when A is a normal subgroup of one of G_j .

It is natural to ask whether it is really necessarily to assume in Proposition 7.6 the existence of a norm N_G such that $N_A \leq N_G$. The following example shows that this assumption cannot be dropped.

Example. Let G be the discrete Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

and let A be the center of G

$$A = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

The subgroup A is, of course, isomorphic to the group of integers \mathbb{Z} . Let d be a metric on A given by the absolute value: $d(b_1, b_2) = |b_1 - b_2|$. We claim that this metric can not be extended to a tsi (in fact, even to a left-invariant) metric on G . Indeed, suppose there is such an extension \underline{d} . The group G is generated by the three matrices:

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and } z = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $z^{n^2} = [x^n, y^n] = x^n y^n x^{-n} y^{-n}$. Therefore

$$n^2 = d(z^{n^2}, e) = \underline{d}(z^{n^2}, e) = \underline{d}(x^n y^n x^{-n} y^{-n}, e) \leq 2n(\underline{d}(x, e) + \underline{d}(y, e)),$$

for all n , which is an absurd.

8. INDUCED METRICS

In this section (G, d) denotes a tsi group, and $A < G$ is a closed subgroup. This section is a preparation for the HNN construction, which is given in the next section. Let $\langle t \rangle$ denote a copy of the free group on one element t , i.e., a copy of the integers, with the usual metric $d(t^m, t^n) = |m - n|$. The Grave metric on the free product $G * \langle t \rangle$ is denoted again by the letter d . Consider the subgroup of the free product generated by G and tAt^{-1} ; it not hard to check that, in fact, as an abstract group it is isomorphic to the free product $G * tAt^{-1}$. Thus we have two

metrics on the group $G * tAt^{-1}$: one is just the metric d , the other one is the Graev metric on this free product; denote the latter by \underline{d} . When these two metrics are the same? It turns out that they are the same if and only if the diameter of A is at most 1. The proof of this fact is the core of this section.

We can naturally view $\text{Words}(G \cup tAt^{-1})$ as a subset of $\text{Words}(G \cup \langle t \rangle)$ by treating a letter $tat^{-1} \in tAt^{-1}$ as a word $t \frown a \frown t^{-1} \in \text{Words}(G \cup \langle t \rangle)$. In what follows we identify $\text{Words}(G \cup tAt^{-1})$ with a subset of $\text{Words}(G \cup \langle t \rangle)$.

Let $f \in G * tAt^{-1}$ be given and let $\alpha \in \text{Words}(G \cup tAt^{-1})$ be the reduced form of f . Note that since we have a free product (no amalgamation), reduced form is unique. The word $\alpha \in \text{Words}(G \cup \langle t \rangle)$ can be written as

$$\alpha = g_1 \frown t \frown a_1 \frown t^{-1} g_2 \frown t \frown a_2 \frown t^{-1} \frown \dots \frown t \frown a_n \frown t^{-1} \frown g_{n+1},$$

where $g_i \in G$, $a_i \in A$, and also g_1 or g_{n+1} may be absent.

Lemma 5.8 implies

$$d(f, e) = \inf\{\rho(\alpha, \zeta) : (\alpha, \zeta) \text{ is a congruent } f\text{-pair}\},$$

and notice that the infimum is taken over all pairs with the same first coordinate α — the reduced form of f . We can also impose some restrictions on ζ and change the infimum to a minimum, but we do not need this for a moment.

In the rest of the section ζ, ξ, δ denote words in the alphabet $G \cup \langle t \rangle$.

8.1. Hereditary words.

Definition 8.1. A trivial word $\zeta \in \text{Words}(G \cup \langle t \rangle)$ is called *hereditary* if $\zeta(i) \in \langle t \rangle \setminus \{e\}$ implies $\zeta(i) = t^{\pm 1}$ for all $i \in [1, n]$. A congruent f -pair (α, ζ) , where $f \in G * tAt^{-1}$, is called *hereditary* if α is the reduced form of f , ζ is hereditary, and moreover,

$$\zeta(i) = t^{\pm 1} \implies \zeta(i) = \alpha(i).$$

Lemma 8.2. Let $f \in G * tAt^{-1}$, and let $\alpha \in \text{Words}(G \cup \langle t \rangle)$ be the reduced form of f . If (α, ζ) is a congruent f -pair, then there exists a trivial word $\xi \in \text{Words}(G \cup \langle t \rangle)$ such that (α, ξ) is a hereditary f -pair and $\rho(\alpha, \xi) \leq \rho(\alpha, \zeta)$.

Proof. Let T_ζ be an evaluation tree for ζ . Fix $t \in T_\zeta$. Suppose there exists $j \in R_t$ such that $\alpha(j) = t^{\pm 1}$ and neither $\zeta(j) = \alpha(j)$ nor $\zeta(j) = e$. Since $\zeta(j) \neq e$ and because the pair (α, ζ) is congruent, it must be the case that $\zeta(j) = t^M$ for some $M \neq 0$. Let $\{i_k\}_{k=1}^m \subseteq R_t$ be the complete list of external letters of ζ in R_t , note that $j \in \{i_k\}_{k=1}^m$. Since R_t is ζ -congruent, we have $\zeta(i_k) \cong t$ for all $k \in [1, m]$. Note that since we have a free product, any evaluation tree is, in fact, slim, and any congruent f -pair is, in fact, a simple f -pair. So we can perform a symmetrization. Set

$$\delta = \text{Sym}(\alpha, \zeta; i_1, \{i_k\}).$$

By Lemma 5.6 $\rho(\alpha, \delta) \leq \rho(\alpha, \zeta)$ and also for all $i \in R_t$ we have

$$\alpha(i) = t^{\pm 1} \implies (\alpha(i) = \delta(i)) \text{ or } (\delta(i) = e) \text{ or } (i = i_1).$$

Let $\epsilon_k \in \{-1, +1\}$ be such that $\alpha(i_k) = t^{\epsilon_k}$. For all $k \in [2, m]$

$$\delta(i_k) = \alpha(i_k) = t^{\epsilon_k}.$$

Let N be such that $\delta(i_1) = t^N$. Note that since $\hat{\delta}[I_t] = e$,

$$N + \epsilon_2 + \dots + \epsilon_m = 0.$$

We now construct a word $\bar{\xi}$ as follows.

Case 0. If $N = 0$ or $N = \epsilon_1$, then set $\bar{\xi} = \delta$.

In cases below we assume $N \notin \{0, \epsilon_1\}$.

Case 1. Suppose $\text{sign}(N) = \text{sign}(\epsilon_1)$. Find different indices $k_1, \dots, k_{|N|-1}$ such that $\text{sign}(N) = -\text{sign}(\epsilon_{k_p})$ for all $p \in [1, |N| - 1]$. Set

$$\bar{\xi}(i) = \begin{cases} \delta(i) & \text{if } i \notin \{i_{k_p}\}_{p=1}^{|N|-1} \text{ and } i \neq i_1; \\ \alpha(i_1) & \text{if } i = i_1; \\ e & \text{if } i \in \{i_{k_p}\}_{p=1}^{|N|-1}. \end{cases}$$

Case 2. Suppose $\text{sign}(N) = -\text{sign}(\epsilon_1)$. Find different indices $k_1, \dots, k_{|N|}$ such that $\text{sign}(N) = -\text{sign}(\epsilon_{k_p})$ for all $p \in [1, |N|]$. Set

$$\bar{\xi}(i) = \begin{cases} \delta(i) & \text{if } i \notin \{i_{k_p}\}_{p=1}^{|N|} \text{ and } i \neq i_1; \\ e & \text{if } i \in \{i_{k_p}\}_{p=1}^{|N|} \text{ or } i = i_1. \end{cases}$$

It is easy to check that $\rho(\alpha, \delta) = \rho(\alpha, \bar{\xi})$ and $\hat{\xi} = e$. Moreover, for all $i \in R_t$ if $\alpha(i) = t^{\pm 1}$, then either $\bar{\xi}(i) = \alpha(i)$ or $\bar{\xi}(i) = e$.

Now apply the same procedure for all $t \in T_\zeta$ and denote the result by ξ . The word ξ is as desired. \square

To analyze the structure of hereditary words we introduce the following notion of a structure tree.

Definition 8.3. Let ζ be a hereditary word of length n . A tree T_ζ together with a function that assigns to a node $t \in T_\zeta$ an interval $I_t \subseteq [1, n]$ is called a *structure tree* for ζ if for all $s, t \in T_\zeta$ the following conditions are met:

- (i) $I_\emptyset = [1, n]$;
- (ii) $\hat{\zeta}[I_t] = e$;
- (iii) if $t \neq \emptyset$, then $\zeta(m(I_t)) = t^{\pm 1}$ and $\zeta(M(I_t)) = t^{\mp 1}$ (in particular $\zeta(m(I_t)) = \zeta(M(I_t))^{-1}$).

Set $R_t = I_t \setminus \bigcup_{t' \prec t} I_{t'}$; then also

- (v) for all $i \in R_t$ if $i \notin \{m(I_t), M(I_t)\}$, then $\zeta(i) \in G$ (in particular $R_t \setminus \{m(I_t), M(I_t)\}$ is ζ -congruent);
- (vi) $\zeta(i) \in G$ for all $i \in R_\emptyset$ (in general R_\emptyset may be empty);
- (vii) if $H(t) \leq H(s)$ and $I_s \cap I_t \neq \emptyset$, then $s \prec t$ or $s = t$;
- (viii) if $s \prec t$ and $t \neq \emptyset$, then

$$m(I_t) < m(I_s) < M(I_s) < M(I_t).$$

Lemma 8.4. If ζ is a hereditary word of length n , then

$$|\{i \in [1, n] : \zeta(i) = t\}| = |\{i \in [1, n] : \zeta(i) = t^{-1}\}|.$$

Proof. Let $\{i_k\}_{k=1}^m$ be the list of letters such that

- (i) $\zeta(i_k) = t^{\epsilon_k}$ for some $\epsilon_k \in \{-1, 1\}$;
- (ii) $\zeta(i) = t^\epsilon$, $\epsilon \in \{-1, 1\}$, implies $i = i_k$ for some k .

Since $\hat{\zeta} = e$, we get

$$\epsilon_1 + \dots + \epsilon_m = 0,$$

and therefore

$$|\{i \in [1, n] : \zeta(i) = t\}| = |\{i \in [1, n] : \zeta(i) = t^{-1}\}|. \quad \square$$

Lemma 8.5. *Let ζ be a hereditary word of length n . If there is $i \in [1, n]$ such that $\zeta(i) = t$, then there is an interval $I \subseteq [1, n]$ such that*

- (i) $\zeta(m(I)) = t^{\pm 1}$ and $\zeta(M(I)) = t^{\mp 1}$;
- (ii) $\zeta(i) \in G$ for all $i \in I \setminus \{m(I), M(I)\}$;
- (iii) $\hat{\zeta}[I] = e$.

Proof. Let I_1, \dots, I_m be the list of intervals such that

- (i) $\zeta(m(I_k)) = t^{\pm 1}$, $\zeta(M(I_k)) = t^{\mp 1}$;
- (ii) $\zeta(i) \in G$ for all $i \in I_k \setminus \{m(I_k), M(I_k)\}$;
- (iii) $M(I_k) \leq m(I_{k+1})$;
- (iv) if I is an interval that satisfies (i) and (ii) above, then $I = I_k$ for some $k \in [1, m]$.

It follows from Lemma 8.4 that the list of such intervals is nonempty. Let J_0, \dots, J_m be the complementary intervals:

$$J_0 = [1, m(J_1) - 1], \quad J_m = [M(J_m) + 1, n],$$

$$J_k = [M(I_k) + 1, m(I_{k+1}) + 1] \quad \text{for } k \in [2, m-1].$$

Some (and even all) of the intervals J_k may be empty. If for some $j_1, j_2 \in J_k$ we have $\zeta(j_1) = t^{\epsilon_1}$, $\zeta(j_2) = t^{\epsilon_2}$, then $\epsilon_1 = \epsilon_2$, and moreover, $M(I_k) = \zeta(j_1) = m(I_{k+1})$. It is now easy to see that $\hat{\zeta}[I_k] \neq e$ for all $k \in [1, m]$ implies $\hat{\zeta} \neq e$, contradicting the assumption that ζ is trivial. \square

Lemma 8.6. *If ζ is a hereditary word of length n , then there is a structure tree T_ζ for ζ .*

Proof. We prove the lemma by induction on $|\{i \in [1, n] : \zeta(i) = t\}|$. For the base of induction suppose that $\zeta(i) \neq t$ for all i . By the definition of a hereditary word and by Lemma 8.4 we have $\zeta(i) \in G$ for all $i \in [1, n]$. Set $T_\zeta = \{\emptyset\}$ and $I_\emptyset = [1, n]$. It is easy to see that this gives a structure tree.

Suppose now there is $i \in [1, n]$ such that $\zeta(i) = t$. Apply Lemma 8.5 and let I denote an interval granted by this lemma. Let m be the length of I . If $m = n$, that is if $I = [1, n]$, then set $T_\zeta = \{\emptyset, t\}$ with $t \prec \emptyset$ and $I_t = I_\emptyset = [1, n]$. One checks that this is a structure tree. Assume now that $m < n$. Define a word δ of length $n - m$ by

$$\delta(i) = \begin{cases} \zeta(i) & \text{if } i < m(I) \\ \zeta(i + m) & \text{if } i \geq m(I). \end{cases}$$

The word δ is a hereditary word and

$$|\{i \in [1, |\delta|] : \delta(i) = t\}| < |\{i \in [1, n] : \zeta(i) = t\}|.$$

Therefore, by induction hypothesis, there is a structure tree T_δ and intervals J_t , $t \in T_\delta$, for the word δ . Let s be a symbol for a new node. Set $T_\zeta = T_\delta \cup \{s\}$. If $m(I) = 1$ or $M(I) = n$, set $(s, \emptyset) \in E(T_\delta)$. Otherwise let $t \in T_\delta$ be the minimal node such that $m(J_t) < m(I) \leq M(J_t)$ (t may still be the root \emptyset) and set $(s, t) \in E(T_\delta)$. Finally, define for $t \in T_\delta$

$$I_t = \begin{cases} J_t & \text{if } M(J_t) < m(I); \\ [m(J_t), M(J_t) + m] & \text{if } m(J_t) \leq m(I) \leq M(J_t); \\ [M(J_t) + m, M(J_t) + m] & \text{if } m(I) < m(J_t). \end{cases}$$

and set $I_s = I$.

It is now straightforward to check that T_ζ is a structure tree for ζ . \square

8.2. From hereditary to rigid words. *From now on A will denote a closed subgroup of G of diameter $\text{diam}(A) \leq 1$, unless stated otherwise.*

Lemma 8.7. *If (G, d) is a tsi group, then for all $g_1, \dots, g_{n-1} \in G$, for all $a_1, \dots, a_n \in A$ such that $d(a_i, e) \leq 1$*

$$d(g_1 \cdots g_{n-1}, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) \leq n$$

Proof. By induction. For $n = 2$ we have

$$d(g_1, a_1 g_1 a_2) \leq d(g_1, a_1 g_1) + d(a_1 g_1, a_1 g_1 a_2) = d(e, a_1) + d(e, a_2) \leq 2.$$

For the step of induction

$$\begin{aligned} & d(g_1 \cdots g_{n-1}, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) \leq \\ & d(g_1 \cdots g_{n-1}, g_1 \cdots g_{n-1} a_n) + d(g_1 \cdots g_{n-1} a_n, a_1 g_1 a_2 \cdots a_{n-1} g_{n-1} a_n) = \\ & d(e, a_n) + d(g_1 \cdots g_{n-2}, a_1 g_1 a_2 \cdots g_{n-2} a_{n-1}) \leq 1 + (n-1) = n. \end{aligned}$$

And the lemma follows. \square

Let β be a word of the form

$$\beta = g_0 \frown t \frown a_1 \frown t^{-1} \frown g_1 \frown t \frown a_2 \frown t^{-1} \frown \cdots \frown g_{n-1} \frown t \frown a_n \frown t^{-1} \frown g_n,$$

where $g_i \in G$ and $a_i \in A$.

Define a word δ by setting for $i \in [1, |\beta|]$

$$\delta(i) = \begin{cases} e & \text{if } i \equiv 1 \pmod{4}; \\ t & \text{if } i \equiv 2 \pmod{4}; \\ e & \text{if } i \equiv 3 \pmod{4}; \\ t^{-1} & \text{if } i \equiv 0 \pmod{4}. \end{cases}$$

Or, equivalently,

$$\delta = e \frown t \frown e \frown t^{-1} \frown e \frown \cdots \frown e \frown t \frown e \frown t^{-1} \frown e.$$

Set $\xi = \text{Sym}(\beta, \delta; 1, \{4k+1\}_{k=0}^n)$.

Lemma 8.8. *Let β, ξ be as above. If ζ is a trivial word of length $|\beta|$ that is congruent to β and such that $\zeta(i) \in G$ for all i , in other words if*

$$\zeta = h_0 \frown e \frown h_1 \frown e \frown h_2 \frown e \frown h_3 \frown e \frown \cdots \frown h_{2n-2} \frown e \frown h_{2n-1} \frown e \frown h_{2n},$$

where $h_i \in G$, then $\rho(\beta, \xi) \leq \rho(\beta, \zeta)$.

Proof. By the two-sided invariance

$$\rho(\beta, \zeta) \geq d(g_0 a_1 g_1 a_2 \cdots g_{n-1} a_n g_n, e) + 2n.$$

On the other hand

$$\begin{aligned} \rho(\beta, \xi) &= \sum_{i=1}^n d(a_i, e) + d(g_0 g_1 \cdots g_n, e) \leq \\ & n + d(g_0 g_1 \cdots g_n, e) \leq \\ & n + d(g_0 g_1 \cdots g_n, g_0 a_1 g_1 a_2 \cdots g_{n-1} a_n g_n) + d(g_0 a_1 g_1 a_2 \cdots g_{n-1} a_n g_n, e) = \\ & n + d(g_1 \cdots g_{n-1}, a_1 g_1 \cdots g_{n-1} a_n) + d(g_0 a_1 g_1 a_2 \cdots g_{n-1} a_n g_n, e) \leq \\ & \text{[by Lemma 8.7]} \quad 2n + d(g_0 a_1 g_1 a_2 \cdots g_{n-1} a_n g_n, e). \end{aligned}$$

Hence $\rho(\beta, \xi) \leq \rho(\beta, \zeta)$. \square

Suppose we have words

$$\begin{aligned}\nu_k &= g_{(k,1)} \frown \cdots \frown g_{(k,q_k)}, \quad \text{where } g_{(k,j)} \in G \text{ and } k \in [0, n], \\ \mu_k &= a_{(k,1)} \frown \cdots \frown a_{(k,p_k)}, \quad \text{where } a_{(k,j)} \in A \text{ and } k \in [1, n].\end{aligned}$$

And let $\bar{\beta}$ be the word

$$\bar{\beta} = \nu_0 \frown t \frown \mu_1 \frown t^{-1} \nu_1 \frown \cdots \frown \nu_{n-1} \frown t \frown \mu_n \frown t^{-1} \nu_n.$$

Let $\{i_k\}_{k=1}^n, \{i'_k\}_{k=1}^n$ be indices such that

- (i) $i_k < i_{k+1}, i'_k < i'_{k+1}$;
- (ii) $\beta(i_k) = t, \beta(i'_k) = t^{-1}$;
- (iii) if $\beta(i) = t$, then $i = i_k$ for some $k \in [1, n]$; if $\beta(i) = t^{-1}$, then $i = i'_k$ for some $k \in [1, n]$.

In other words

$$i_k = \sum_{l=0}^{k-1} q_l + \sum_{l=1}^{k-1} p_k + 2(k-1) + 1, \quad i'_k = i_k + p_k + 1.$$

Define the word δ of length $|\bar{\beta}|$ by

$$\delta(i) = \begin{cases} e & \text{if } \bar{\beta}(i) \in G; \\ \bar{\beta}(i) & \text{if } \bar{\beta}(i) = t^{\pm 1}. \end{cases}$$

Let $\{j_k\}_{k=1}^n$ be the enumeration of the set

$$[1, |\bar{\beta}|] \setminus \bigcup_{k=1}^n [i_k, i'_k].$$

Set inductively

$$\begin{aligned}\xi_0 &= \text{Sym}(\bar{\beta}, \delta; j_1, \{j_k\}), \\ \xi_{l+1} &= \text{Sym}(\bar{\beta}, \xi_l; j_1^{(l+1)}, \{j_k^{(l+1)}\}),\end{aligned}$$

where $j_k^{(l)} = i_l + k, l \in [1, n], k \in [1, p_k]$. Finally set $\bar{\xi} = \xi_n$.

Example. For example, if

$$\bar{\beta} = g_1 \frown g_2 \frown t \frown a_1 \frown a_2 \frown a_3 \frown t^{-1} \frown g_3,$$

then

$$\begin{aligned}\delta &= e \frown e \frown t \frown e \frown e \frown e \frown t^{-1} \frown e, \\ \xi_0 &= x \frown g_2 \frown t \frown e \frown e \frown e \frown t^{-1} \frown g_3, \quad x = g_3^{-1} g_2^{-1}, \\ \xi_1 &= x \frown g_2 \frown t \frown y \frown a_2 \frown a_3 t^{-1} \frown g_3, \quad y = a_3^{-1} a_2^{-1}.\end{aligned}$$

Lemma 8.9. *Let $\bar{\beta}, \bar{\xi}$ be as above. If ζ is a trivial word of length $|\bar{\beta}|$ that is congruent to $\bar{\beta}$ and such that $\zeta(i) \in G$ for all i , then $\rho(\bar{\beta}, \bar{\xi}) \leq \rho(\bar{\beta}, \zeta)$.*

Proof. Set

$$\begin{aligned}\beta &= \hat{\nu}_0 \frown t \frown \hat{\mu}_1 \frown t^{-1} \frown \cdots \frown \hat{\mu}_n \frown t^{-1} \frown \hat{\nu}_n, \\ \xi' &= \hat{\xi}[1, i_1 - 1] \frown t \frown \hat{\xi}[i_1 + 1, i'_1 - 1] \frown t^{-1} \frown \cdots \frown \hat{\xi}[i_n + 1, i'_n - 1] \frown t^{-1} \frown \hat{\xi}[i'_n + 1, n], \\ \zeta' &= \hat{\zeta}[1, i_1 - 1] \frown t \frown \hat{\zeta}[i_1 + 1, i'_1 - 1] \frown t^{-1} \frown \cdots \frown \hat{\zeta}[i_n + 1, i'_n - 1] \frown t^{-1} \frown \hat{\zeta}[i'_n + 1, n].\end{aligned}$$

If ξ is as in Lemma 8.8, then $\xi' = \xi$ and

$$\rho(\bar{\beta}, \zeta) \geq [\text{by tsi}] \rho(\beta, \zeta') \geq [\text{by Lemma 8.8}] \rho(\beta, \xi) = \rho(\beta, \xi') = \rho(\bar{\beta}, \bar{\xi}). \quad \square$$

Let γ be a word of the form

$$\gamma = a_0 \frown t^{-1} \frown g_0 \frown t \frown a_1 \frown t^{-1} \frown g_1 \frown t \frown \cdots \frown a_{n-1} \frown t^{-1} \frown g_{n-1} \frown t \frown a_n,$$

where $g_i \in G$ and $a_i \in A$. Let ζ be a trivial word of the same length that is congruent to γ and such that $\zeta(i) \in G$ for all i . In other words

$$\zeta = h_0 \frown e \frown h_1 \frown e \frown h_2 \frown e \frown h_3 \frown e \frown \cdots \frown h_{2n-2} \frown e \frown h_{2n-1} \frown e \frown h_{2n},$$

where $h_i \in G$. Define a word δ by

$$\delta(i) = \begin{cases} a_0 & \text{if } i = 1; \\ e & \text{if } i = 1 \pmod{4} \text{ and } 1 < i < 4n + 1; \\ t & \text{if } i = 2 \pmod{4}; \\ e & \text{if } i = 3 \pmod{4}; \\ t^{-1} & \text{if } i = 0 \pmod{4}; \\ a_0^{-1} & \text{if } i = 4n + 1. \end{cases}$$

Or, equivalently,

$$\delta = a_0 \frown t \frown e \frown t^{-1} \frown e \frown \cdots \frown e \frown t \frown e \frown t^{-1} \frown a_0^{-1}.$$

Set $\xi = \text{Sym}(\gamma, \delta; 3, \{4k - 1\}_{k=1}^n)$.

Example. For example, if

$$\gamma = a_0 \frown t^{-1} \frown g_0 \frown t \frown a_1 \frown t^{-1} \frown g_1 \frown t \frown a_2,$$

then

$$\begin{aligned} \delta &= a_0 \frown t^{-1} \frown e \frown t \frown e \frown t^{-1} \frown e \frown t \frown a_0^{-1}, \\ \xi &= a_0 \frown t^{-1} \frown g_1^{-1} \frown t \frown e \frown t^{-1} \frown g_1 \frown t \frown a_0^{-1}. \end{aligned}$$

Lemma 8.10. *If γ, ζ, ξ are as above, then $\rho(\gamma, \xi) \leq \rho(\gamma, \zeta)$.*

Proof. By the two-sided invariance

$$\rho(\gamma, \zeta) \geq d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) + 2n.$$

On the other hand

$$\begin{aligned} \rho(\gamma, \xi) &= d(a_0 a_n, e) + n - 1 + d(g_0 g_1 \cdots g_n, e) \leq \\ & n + d(g_0 g_1 \cdots g_{n-1}, a_0^{-1} a_n^{-1}) + d(a_0^{-1} a_n^{-1}, e) \\ & n + 1 + d(g_0 g_1 \cdots g_{n-1}, a_0^{-1} a_n^{-1}) \leq \\ & n + 1 + d(a_0 g_0 g_1 \cdots g_{n-2} g_{n-1} a_n, a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n) + \\ & d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) = \\ & n + 1 + d(g_1 \cdots g_{n-2}, a_1 g_1 \cdots g_{n-2} a_{n-1}) + \\ & d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) \leq [\text{by Lemma 8.7}] \\ & n + 1 + n - 1 + d(a_0 g_0 a_1 g_1 \cdots a_{n-1} g_{n-1} a_n, e) \leq \rho(\gamma, \zeta). \end{aligned}$$

And the lemma follows. \square

Suppose we have words

$$\begin{aligned}\mu_k &= a_{(k,1)} \frown \cdots \frown a_{(k,p_k)}, & \text{where } a_{(k,j)} \in A \text{ and } k \in [0, n], \\ \nu_k &= g_{(k,1)} \frown \cdots \frown g_{(k,q_k)}, & \text{where } g_{(k,j)} \in G \text{ and } k \in [1, n],\end{aligned}$$

and let $\bar{\gamma}$ be the word

$$\bar{\gamma} = \mu_0 \frown t^{-1} \frown \nu_0 \frown t \frown \mu_1 \frown \cdots \frown \mu_{n-1} \frown t^{-1} \frown \nu_{n-1} \frown t \frown \mu_n.$$

Let $\{i_k\}_{k=1}^n, \{i'_k\}_{k=1}^n$ be indices such that

- (i) $i_k < i_{k+1}, i'_k < i'_{k+1}$;
- (ii) $\gamma(i_k) = t^{-1}, \gamma(i'_k) = t$;
- (iii) if $\gamma(i) = t^{-1}$, then $i = i_k$ for some $k \in [1, n]$; if $\gamma(i) = t$, then $i = i'_k$ for some $k \in [1, n]$.

Define the word δ of length $|\bar{\gamma}|$ by

$$\delta(i) = \begin{cases} e & \text{if } \bar{\gamma}(i) \in G; \\ \bar{\gamma}(i) & \text{if } \bar{\gamma}(i) = t^{\pm 1}. \end{cases}$$

Let $\{j_k\}_{k=1}^m$ be the enumeration of the set

$$\bigcup_{k=1}^n [i_k + 1, i'_k - 1].$$

Set inductively

$$\begin{aligned}\xi_0 &= \text{Sym}(\bar{\gamma}, \delta; j_1, \{j_k\}), \\ \xi_{l+1} &= \text{Sym}(\bar{\gamma}, \xi_l; j_1^{(l+1)}, \{j_k^{(l+1)}\}),\end{aligned}$$

where $j_k^{(l)} = i'_l + k$ and $l \in [1, n], k \in [1, p_k]$. Finally set

$$\bar{\xi} = \text{Sym}(\bar{\gamma}, \xi_n; 1, [1, i_1 - 1] \cup [i'_n + 1, n]).$$

Example. For example, if

$$\bar{\gamma} = a_1 \frown a_2 \frown t^{-1} \frown g_1 \frown t \frown a_3 \frown a_4 \frown t^{-1} \frown g_2 \frown g_3 \frown t \frown a_5,$$

then

$$\begin{aligned}\delta &= e \frown e \frown t^{-1} \frown e \frown t \frown e \frown e \frown t^{-1} \frown e \frown e \frown t \frown e, \\ \xi_0 &= e \frown e \frown t^{-1} \frown x \frown t \frown e \frown e \frown t^{-1} \frown g_2 \frown g_3 \frown t \frown e, \quad x = g_3^{-1} g_2^{-1} \\ \xi_1 &= e \frown e \frown t^{-1} \frown x \frown t \frown a_4^{-1} \frown a_4 \frown t^{-1} \frown g_2 \frown g_3 \frown t \frown e, \\ \xi &= y \frown a_2 \frown t^{-1} \frown x \frown t \frown a_4^{-1} \frown a_4 \frown t^{-1} \frown g_2 \frown g_3 \frown t \frown a_5, \quad y = a_5^{-1} a_2^{-1}.\end{aligned}$$

Lemma 8.11. *Let $\bar{\gamma}, \bar{\xi}$ be as above. If ζ is a trivial word of length $|\bar{\gamma}|$ that is congruent to $\bar{\gamma}$ and such that $\zeta(i) \in G$ for all i , then $\rho(\bar{\gamma}, \bar{\xi}) \leq \rho(\bar{\gamma}, \zeta)$.*

Proof. Proof is similar to the proof of Lemma 8.9. \square

Definition 8.12. Let (α, ζ) be a hereditary f -pair of length n . It is called *rigid* if for all $i \in [1, n]$

$$\alpha(i) = t^{\pm 1} \implies \zeta(i) = \alpha(i).$$

Here is an example of a rigid pair:

$$\begin{aligned}\alpha &= g_0 \frown t \frown a_1 \frown t^{-1} \frown g_1 \frown t \frown a_2 \frown t^{-1} \frown g_2, \\ \zeta &= g_2^{-1} g_1^{-1} \frown t \frown e \frown t^{-1} \frown g_1 \frown t \frown e \frown t^{-1} \frown g_2.\end{aligned}$$

Lemma 8.13. *Let $f \in G * tAt^{-1}$, and let $\alpha \in \text{Words}(G \cup \langle t \rangle)$ be the reduced form of f . If (α, ζ) is a hereditary f -pair, then there exists a rigid f -pair (α, ξ) such that $\rho(\alpha, \zeta) \geq \rho(\alpha, \xi)$. Moreover, if for some i one has $\alpha(i) = t$, then $\xi(i+1) \in A$.*

Proof. Let (α, ζ) be hereditary and let T_ζ be a structure tree for ζ . If $t \in T_\zeta$ and $Q_t = R_t \setminus \{m(I_t), M(I_t)\}$, then, depending on whether $\zeta(m(I_t)) = t^{-1}$ or $\zeta(m(I_t)) = t$, we have

$$\begin{aligned} \alpha[Q_t] = \bar{\beta} &= g_{(0,1)} \frown \cdots \frown g_{(0,q_0)} \frown t \frown a_{(1,1)} \frown \cdots \frown a_{(1,p_1)} \frown t^{-1} \frown \cdots \\ &\quad \cdots \frown t \frown a_{(n,1)} \frown \cdots \frown a_{(n,p_n)} \frown t^{-1} \frown g_{(n,1)} \cdots g_{(n,q_n)}, \end{aligned}$$

or

$$\begin{aligned} \alpha[Q_t] = \bar{\gamma} &= a_{(0,1)} \frown \cdots \frown a_{(0,p_0)} \frown t^{-1} \frown g_{(0,1)} \frown \cdots \frown g_{(0,q_1)} \frown t \frown \cdots \\ &\quad \cdots \frown t^{-1} \frown g_{(n-1,1)} \frown \cdots \frown g_{(n-1,q_n)} \frown t \frown a_{(n,1)} \cdots a_{(n,p_n)}, \end{aligned}$$

where $a_{(i,j)} \in A$ and $g_{(i,j)} \in G$.

Let $\bar{\xi}_t$ be as in Lemma 8.9 or in Lemma 8.11 depending on whether $\alpha[Q_t] = \bar{\beta}$ or $\alpha[Q_t] = \bar{\gamma}$ and set

$$\begin{aligned} \xi[Q_t] &:= \bar{\xi}_t, \quad \xi(m(I_t)) = \alpha(m(I_t)), \quad \xi(M(I_t)) = \alpha(M(I_t)) \quad \text{if } t \neq \emptyset, \\ \xi[R_\emptyset] &:= \bar{\xi}_\emptyset, \quad \text{if } t = \emptyset. \end{aligned}$$

Do this for all $t \in T_\zeta$. Then (α, ξ) is a rigid f -pair and

$$\rho(\alpha, \zeta) \geq \rho(\alpha, \xi) \quad [\text{by Lemma 8.9 and Lemma 8.11}].$$

The moreover part follows immediately from the construction of ξ . \square

Theorem 8.14. *Let (G, d) be a tsi group, $A < G$ be a closed subgroup, not necessarily of diameter at most one. If d and \underline{d} are as before (see the beginning of Section 8), then $d = \underline{d}$ if and only if $\text{diam}(A) \leq 1$.*

Proof. First we show that the condition $\text{diam}(A) \leq 1$ is necessary. Suppose $\text{diam}(A) > 1$ and let $a \in A$ be such that $d(a, e) > 1$. Then

$$\begin{aligned} \underline{d}(ata^{-1}t^{-1}, e) &= d(a, e) + d(ta^{-1}t^{-1}, e) = d(a, e) + d(a^{-1}, e) = 2d(a, e) > 2, \\ d(ata^{-1}t^{-1}, e) &= d(ata^{-1}t^{-1}, aea^{-1}e) \leq \\ d(a, a) + d(t, e) + d(a^{-1}, a^{-1}) + d(t^{-1}, e) &= 2. \end{aligned}$$

And so $\underline{d} \neq d$.

Suppose now $\text{diam}(A) \leq 1$. Let $f \in G * tAt^{-1}$ be given and let α be the reduced form of f . If (α, ζ) is a congruent f -pair, then by Lemma 8.2 and Lemma 8.13 there is a rigid f -pair (α, ξ) such that $\rho(\alpha, \xi) \leq \rho(\alpha, \zeta)$ and $\alpha(i) = t$ implies $\xi(i+1) \in A$. Hence we can view ξ as an element in $\text{Words}(G \cup tAt^{-1})$. Since ζ was arbitrary, it follows that $\underline{d}(f, e) \leq d(f, e)$. The inverse inequality $d(f, e) \leq \underline{d}(f, e)$ follows from item (iii) of Proposition 6.5. Thus $\underline{d}(f, e) = d(f, e)$, and, by the left invariance, $\underline{d}(f_1, f_2) = d(f_1, f_2)$ for all $f_1, f_2 \in G * tAt^{-1}$. \square

Proposition 8.15. *Let (G, d) be a tsi group, $A < G$ be a subgroup and \underline{d} be the Graev metric on the free product $G * \langle t \rangle$. We can naturally view $G * tAt^{-1}$ as a subgroup of $G * \langle t \rangle$. If A is closed in G , then $G * tAt^{-1}$ is closed in $G * \langle t \rangle$.*

Proof. The proof is similar in spirit to the proof of item (ii) of Proposition 6.5, but requires some additional work. Suppose the statement is false and there is $f \in G * \langle t \rangle$ such that $f \notin G * tAt^{-1}$, but $f \in \overline{G * tAt^{-1}}$. Let $\alpha \in \text{Words}(G \cup \langle t \rangle)$ be the reduced form of f , $n = |\alpha|$. We show that this is impossible and $f \in G * tAt^{-1}$. The proof goes by induction on n .

Base of induction. For the base of induction we consider cases $n \in \{1, 2\}$. If $n = 1$, then either $f \in G$ or $f = t^k$ for some $k \neq 0$. Since $G < G * tAt^{-1}$, it must be the case that $f = t^k$. Let $h \in G * tAt^{-1}$ be such that $\underline{d}(f, h) < 1$, where \underline{d} is the Graev metric on $G * \langle t \rangle$. Let $\phi_1 : G \rightarrow \mathbb{Z}$ be the trivial homomorphism: $\phi_1(g) = 0$ for all $g \in G$; and let $\phi_2 : \langle t \rangle \rightarrow \mathbb{Z}$ be the natural isomorphism: $\phi_2(t^k) = k$. By item (i) of Proposition 6.5 ϕ_1 and ϕ_2 extend to a 1-Lipschitz homomorphism $\phi : G * \langle t \rangle \rightarrow \mathbb{Z}$. But $d(\phi(f), \phi(h)) = |k| \geq 1$. We get a contradiction with the assumption $\underline{d}(f, h) < 1$.

Note that for any $h \in G * tAt^{-1}$

$$f \in \left(\overline{G * tAt^{-1}} \right) \setminus G * tAt^{-1} \implies gh, hg \in \left(\overline{G * tAt^{-1}} \right) \setminus G * tAt^{-1}.$$

Using this observation the case $n = 2$ follows from the case $n = 1$. Indeed, $n = 2$ implies $\alpha = g \frown t^k$ or $\alpha = t^k \frown g$ for some $g \in G$, $k \neq 0$. Multiplying f by g from either left or right brings us to the case $n = 1$.

Step of induction. Without loss of generality we may assume that $\alpha(n) = t^k$ for some $k \neq 0$. Indeed, if $\alpha(n) = g$ for some $g \in G$, then we can substitute fg^{-1} for f . Assume that $\alpha = \alpha_0 \frown t^{k_1} \frown g \frown t^{k_2}$, where $k_1, k_2 \neq 0$ and $g \in G$. We claim that $k_1 = 1$, $k_2 = -1$, and $g \in A$. Set

$$\begin{aligned} \epsilon_1 &= \min\{d(\alpha(i), e) : i \in [1, n]\}, \\ \epsilon_2 &= \begin{cases} 1 & \text{if } \forall i \alpha(i) \in G \implies \alpha(i) \in A, \\ \min\{d(\alpha(i), A) : \alpha(i) \in G \setminus A\} & \text{otherwise.} \end{cases} \end{aligned}$$

And let $\epsilon = \min\{1, \epsilon_1, \epsilon_2\}$. Note that $\epsilon > 0$.

Since $f \in \overline{G * tAt^{-1}}$, there is $h \in G * tAt^{-1}$ such that $\underline{d}(f, h) < \epsilon$. Therefore there is a reduced simple fh^{-1} -pair (β, ξ) such that $\rho(\beta, \xi) < \epsilon$. Let γ be the reduced form of h^{-1} . Suppose first that $k_2 \neq -1$. Assume for simplicity that $\beta = \alpha \frown \gamma$ (in general the first letter of γ may get canceled; the proof for the general case is the same, it is just notationally simpler to assume that $\beta = \alpha \frown \gamma$). Let T_ξ be the slim evaluation tree for ξ , and let $s_0 \in T_\xi$ be such that $n \in R_{s_0}$.

We claim that $n = m(R_{s_0})$. If this is not the case, then there is $i_0 \in R_{s_0}$ such that $i_0 < n$ and $[i_0 + 1, n - 1] \cap R_{s_0} = \emptyset$. Since α is reduced, $i_0 < n - 1$. If $I = [i_0 + 1, n - 1]$, then $\hat{\xi}[I] = e$ and so there is $j_0 \in I$ such that $\xi(j_0) = e$. Therefore

$$\rho(\beta, \xi) \geq d(\beta(j_0), \xi(j_0)) = d(\alpha(j_0), e) \geq \epsilon_1 \geq \epsilon.$$

Contradicting the choice of the pair (β, ξ) .

Thus $n = m(R_{s_0})$. Let j_1, \dots, j_p be such that

- (i) $j_k \in R_{s_0}$ for all $k \in [1, p]$;
- (ii) $j_k < j_{k+1}$;
- (iii) $\xi(j_k) \neq e$;
- (iv) $\xi(j) \neq e$ and $j \in R_{s_0}$ implies $j = j_k$ for some k .

In fact, we can always modify the tree to assure that $\xi(j) \neq e$ for all $j \in R_{s_0}$, but this is not used here. In this notation $j_1 = n$. Since $\rho(\beta, \xi) < 1$, we get

$\beta(j_k) = \xi(j_k) = t^{\pm 1}$ for all $k \in [2, p]$. If $I_k = [j_k + 1, j_{k+1} - 1]$ for $k \in [1, p - 1]$, then $\hat{\xi}[I_k] = e$ for all k , whence for any $k \in [1, p - 1]$

$$|\{i \in I_k : \xi(i) = t\}| = |\{i \in I_k : \xi(i) = t^{-1}\}|.$$

This implies

$$\xi(j_2) = t, \xi(j_3) = t^{-1}, \xi(j_4) = t, \dots, \xi(j_p) = t^{((-1)^p)}.$$

Finally, since $\hat{\xi}[R_{s_0}] = e$, we get $\xi(j_1) = t^{-1}$ or $\xi(j_1) = e$, depending on whether p is even or odd. But since by assumption $k_2 \neq 0$ we get $k_2 = -1$.

We have proved that $k_2 = -1$. The next step is to show that $g \in A$. We have two cases.

Case 1. $\gamma(1) \in G$. In this case we have $\beta = \alpha \frown \gamma$. Let $s_1 \in T_\xi$ be such that $n - 1 \in R_{s_1}$. Similarly to the previous step one shows that $n - 1 = m(R_{s_1})$. Let $R_{s_1} = \{j_k\}_{k=1}^p$, where $j_k < j_{k+1}$. In particular, $n - 1 = j_1$. Set $I_k = [j_k + 1, j_{k+1} - 1]$. From $\hat{\xi}[I_k] = e$ it follows

$$|\{i \in I_k : \xi(i) = t\}| = |\{i \in I_k : \xi(i) = t^{-1}\}|.$$

Therefore $\xi(j_k) \in A$ for all $k \in [2, p]$. And so $\xi(j_1) \in A$ as well. Finally, if $g \notin A$, then

$$\rho(\beta, \xi) \geq d(\beta(n - 1), \xi(n - 1)) \geq d(g, A) \geq \epsilon_2 \geq \epsilon.$$

And again we have a contradiction with the choice of (β, ξ) .

Case 2. $\gamma(1) = t$. In this case $\alpha = \alpha_0 \frown t^{k_1} \frown g \frown t^{-1}$ and $\gamma = t \frown a \frown t^{-1} \frown \gamma_0$, for some $a \in A$ and a word γ_0 . If $g \notin A$ then $\beta = \alpha_0 \frown t^{k_1} \frown ga \frown t^{-1} \frown \gamma_0$. And we are essentially in Case 1. Therefore by the proof of Case 1 we get $ga \in A$, but then $g \in A$.

Thus $g \in A$. The proof of $k_1 = 1$ is similar to the proof of $k_2 = -1$ given earlier, and we omit the details.

We have shown that $\alpha = \alpha_0 \frown t \frown a \frown t^{-1}$. If $f' = fta^{-1}t^{-1}$, then α_0 is the reduced form of f' and $f' \in \overline{G * tAt^{-1}} \setminus G * tAt^{-1}$. We proceed by induction on the length of α . \square

9. HNN EXTENSIONS OF GROUPS WITH TSI METRICS

We now turn to the HNN construction itself. There are several ways to build an HNN extension. We will follow the original construction of G. Higman, B. H. Neumann and H. Neumann from [5], because their approach hides a lot of complications into the amalgamation of groups, and we have already constructed Graev metrics on amalgams in the previous sections.

Let us briefly remind what an HNN extension is. Let G be an abstract group, $A, B < G$ be isomorphic subgroups and $\phi : A \rightarrow B$ be an isomorphism between them. An HNN extension of (G, ϕ) is a pair (H, t) , where t is a new symbol and $H = \langle G, t | tat^{-1} = \phi(a), a \in A \rangle$. The element t is called a *stable letter* of the HNN extension.

9.1. Metrics on HNN extensions.

Theorem 9.1. *Let (G, d) be a tsi group, $\phi : A \rightarrow B$ be a d -isometric isomorphism between the closed subgroups A, B . Let H be the HNN extension of (G, ϕ) in the abstract sense, and let t be the stable letter of the HNN extension. If $\text{diam}(A) \leq K$, then there is a tsi metric \underline{d} on H such that $\underline{d}|_G = d$ and $\underline{d}(t, e) = K$.*

Proof. First assume that $K = 1$. Let $\langle u \rangle$ and $\langle v \rangle$ be two copies of the group \mathbb{Z} of the integers with the usual metric. Form the free products $(G * \langle u \rangle, d_u)$ and $(G * \langle v \rangle, d_v)$, where d_u, d_v are the Graev metrics. Since $\text{diam}(A) = \text{diam}(B) \leq 1$, by Theorem 8.14 the Graev metric on $G * uAu^{-1}$ is the restriction of d_u onto $G * uAu^{-1}$, and, similarly, the Graev metric on $G * vBv^{-1}$ is just the restriction of d_v . Let $\psi : G * uAu^{-1} \rightarrow G * vBv^{-1}$ be an isomorphism that is uniquely defined by

$$\psi(g) = g, \quad \psi(uau^{-1}) = v\phi(a)v^{-1}, \quad a \in A, g \in G.$$

By Theorem 8.14 ψ is an isometry. Also, by Proposition 8.15 $G * uAu^{-1}$ and $G * vBv^{-1}$ are closed subgroups of $G * \langle u \rangle$ and $G * \langle v \rangle$ respectively. Hence by the results of Section 5 we can amalgamate $G * \langle u \rangle$ and $G * \langle v \rangle$ over $G * uAu^{-1} = G * vBv^{-1}$. Denote the result of this amalgamation by $(\tilde{H}, \underline{d})$. Then

$$uau^{-1} = v\phi(a)v^{-1} \quad \text{for all } a \in A,$$

and therefore $v^{-1}uau^{-1}v = \phi(a)$. If $H = \langle G, v^{-1}u \rangle$, then $(H, v^{-1}u)$ is an HNN extension of (G, ϕ) and $\underline{d}|_{H_\phi}$ is a two-sided invariant metric on H , which extends d .

This was done under the assumption that $K = 1$. The general case can be reduced to this one. If $d' = (1/K)d$, then d' is a tsi metric on G , ϕ is a d' -isometric isomorphism and $d' \cdot \text{diam}(A) \leq 1$. By the above construction there is a tsi metric \underline{d}' on H such that $\underline{d}'|_G = d'$. Now set $\underline{d} = K\underline{d}'$. \square

It is, of course, natural to ask if the condition of having a bounded diameter is crucial. The answer to this question is not known, but here is a necessary condition.

Proposition 9.2. *Let (G, d) be a tsi group, $\phi : A \rightarrow B$ be a d -isometric isomorphism, and H be the HNN extension of (G, ϕ) with the stable letter t . If d is extended to a tsi metric d' on H , then*

$$\sup\{d'(a, \phi(a)) : a \in A\} < \infty.$$

Proof. If $K = d'(t, e)$, then for any $a \in A$

$$\begin{aligned} d'(a, \phi(a)) &= d'(a, tat^{-1}) = d'(a^{-1}tat^{-1}, e) = \\ &= d'(a^{-1}tat^{-1}, a^{-1}eae) \leq d'(t, e) + d'(t^{-1}, e) = 2K. \end{aligned}$$

Therefore $\sup\{d'(a, \phi(a)) : a \in A\} \leq 2K$. \square

Question 9.3. *Is this condition also sufficient? To be precise, suppose (G, d) is a tsi group, $\phi : A \rightarrow B$ is a d -isometric isomorphism between closed subgroups A, B , and suppose that*

$$\sup\{d(a, \phi(a)) : a \in A\} < \infty.$$

Does there exist a tsi metric \underline{d} on the HNN extension H of (G, ϕ) such that $\underline{d}|_G = d$?

9.2. Induced conjugation and HNN extension. Recall that a topological group G is called SIN if for every open $U \subseteq G$ such that $e \in U$ there is an open subset $V \subseteq U$ such that $gVg^{-1} = V$ for all $g \in G$. A metrizable group admits a compatible two-sided invariant metric if and only if it is SIN.

Theorem 9.4. *Let G be a SIN metrizable group. Let $\phi : A \rightarrow B$ be a topological isomorphism between two closed subgroups. There exist a SIN metrizable group H and an element $t \in H$ such that $G < H$ is a topological subgroup and $tat^{-1} = \phi(a)$ for all $a \in A$ if and only if there is a compatible tsi metric d on G such that ϕ becomes a d -isometric isomorphisms.*

Proof. Necessity of the condition is obvious: if d is a compatible tsi metric on H , then ϕ is $d|_G$ -isometric. We show sufficiency. Let d be a compatible tsi metric on G such that ϕ is a d -isometric isomorphism. If $d'(g, e) = \min\{d(g, e), 1\}$, then d' is also a compatible tsi metric on G , ϕ is a d' -isometric isomorphism, and d' -diam(A) ≤ 1 (because d' -diam(G) ≤ 1). Apply Theorem 9.1 to get an extension of d' to a tsi metric on H , where (H, t) is the HNN extension of (G, ϕ) . Then (H, t) satisfies the conclusions of the theorem. \square

Corollary 9.5. *Let G be a SIN metrizable group. Let $\phi : A \rightarrow B$ be a topological group isomorphism. If there is a compatible tsi metric d on G such that*

$$\inf\{d(a_1, a_2) : a_1, a_2 \in A, a_1 \neq a_2\} > 0, \quad \inf\{d(b_1, b_2) : b_1, b_2 \in B, b_1 \neq b_2\} > 0,$$

then there is topology on the HNN extension of (G, ϕ) such that G is a closed subgroup of H and H is SIN and metrizable.

Proof. Let d be a compatible tsi metric on G for which there exists a constant $c > 0$ such that

$$\inf\{d(a_1, a_2) : a_1, a_2 \in A\} \geq c, \quad \inf\{d(b_1, b_2) : b_1, b_2 \in B, b_1 \neq b_2\} \geq c.$$

If $d'(g_1, g_2) = \min\{d(g_1, g_2), c\}$, then d' is a compatible tsi metric on G and ϕ is a d' -isometric isomorphism. Theorem 9.4 finishes the proof. \square

Corollary 9.6. *Let $(G, +)$ be an abelian SIN metrizable group. If $\phi : G \rightarrow G$ is given by $\phi(x) = -x$, then there is a SIN metrizable topology on the HNN extension H of (G, ϕ) that extends the topology of G .*

Proof. If d is a compatible tsi metric on G such that d -diam(G) ≤ 1 , then ϕ is a d -isometric isomorphism and we apply Theorem 9.4. \square

Definition 9.7. Let G be a topological group. Elements $g_1, g_2 \in G$ are said to be *induced conjugated* if there exist a topological group H and an element $t \in H$ such that $G < H$ is a topological subgroup and $tg_1t^{-1} = g_2$.

Example 9.8. Let $(\mathbb{T}, +)$ be a circle viewed as a compact abelian group, and let $g_1, g_2 \in \mathbb{T}$. The elements g_1 and g_2 are induced conjugated if and only if one of the two conditions is satisfied:

- (i) g_1 and g_2 are periodic elements of the same period;
- (ii) $g_1 = \pm g_2$.

Proof. The sufficiency of any of these conditions follows from Corollary 9.5 and Corollary 9.6. We need to show the necessity. If g_1 and g_2 are induced conjugated, then they have the same order. If the order of g_i is finite, we are done. Suppose the order is infinite. The groups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are naturally isomorphic (as topological

groups) via the map $\phi(kg_1) = kg_2$. This map extends to a continuous isomorphism $\phi : \mathbb{T} \rightarrow \mathbb{T}$, because \mathbb{T} is compact and $\langle g_i \rangle$ is dense in \mathbb{T} . But there are only two continuous isomorphisms of the circle: $\phi = id$ and $\phi = -id$. Thus $g_1 = \pm g_2$. \square

Example 9.9. Let $G = \mathbb{T}^{\mathbb{Z}}$ be a product of circles, and let $S : \mathbb{T}^{\mathbb{Z}} \rightarrow \mathbb{T}^{\mathbb{Z}}$ be the shift map $S(x)(n) = x(n+1)$ for all $x \in \mathbb{T}^{\mathbb{Z}}$ and all $n \in \mathbb{Z}$. The group $\mathbb{T}^{\mathbb{Z}}$ is monothetic and abelian. If $x = \{a_n\}_{n \in \mathbb{Z}}$, where a_n 's and 1 are linearly independent over \mathbb{Q} , then $\langle x \rangle$ is dense in $\mathbb{T}^{\mathbb{Z}}$. Since S is an automorphism, x and $S(x)$ are topologically similar. We claim that x and $S(x)$ are not induced conjugated in any SIN metrizable group H .

Proof. Suppose H is a SIN metrizable group, G is a topological subgroup of H and $t \in H$ is such that $txt^{-1} = S(x)$. If $\phi_t : H \rightarrow H$ is given by $\phi_t(y) = tyt^{-1}$, then $\phi_t(mx) = S(mx)$ for all $m \in \mathbb{Z}$ and hence, by continuity and density of $\langle x \rangle$, $\phi_t(y) = S(y)$ for all $y \in \mathbb{T}^{\mathbb{Z}}$. If d is a compatible tsi metric on H , then ϕ_t is a d -isometric isomorphism. Therefore for $x_0 \in \mathbb{T}^{\mathbb{Z}}$,

$$x_0(n) = \begin{cases} 1/2 & \text{if } n = 0; \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$d(\phi_t^m(x_0), e) = d(\phi_t^m(x_0), \phi_t^m(e)) = d(x_0, e) = \text{const} > 0,$$

but $S^m(x_0) \rightarrow 0$, when $m \rightarrow \infty$. This contradicts $\phi_t(y) = S(y)$ for all $y \in \mathbb{T}^{\mathbb{Z}}$. \square

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